PROGRESS OF THEORETICAL PHYSICS

(Founded by H. Yukawa in 1946)

Volume 2 Numbers 1～2

January ～ ~~June,~~ 1947
Dec

Published by
The Yukawa Hall

with the cooperation of
The Physical Society of Japan

Progress of Theoretical Physics

A journal of theoretical physics publishing mainly those papers from the fields of quantum mechanics, statistical mechanics and the theory of elementary particles.

Board of Editors

K. Husimi M. Kotani T. Muto
S. Sakata S. Tomonaga H. Yukawa
M. Kobayashi, *Managing Editor.*

A volume of Progress of Theoretical Physics will consist of six numbers each of them published once every other month. Manuscripts offered for publication should be submitted to the managing editor, M. Kobayashi, Department of Physics, Kyoto University. The length of a paper is restricted for the time being within 20 papes. Each number is ordinary provided with a column for letters to the editors containing brief reports of importance in theoretical physics. The length of each letter is restricted within 1000 words. Proof and all correspondence concerning papers in the process of publication should be addressed to the publication office in Department of Physics, Kyoto University.

Price	**$ 2.00**	per copy
	$ 12.00	per year (postage inclusive)

The membership fee (subscription) for the members of the "Progress of Theoretical Physics" Publication Society is **$ 7.50** per year (postage inclusive). Applications for membership, subscriptious of orders for back numbers should be sent, through bookdealers or directly, to

The Publication Office
in
The Yukawa Hall
Kyoto University
Kyoto, Japan

(Subcription may be paid by UNESCO book coupons)

Progress of Theoretical Physics Vol. II, No. 1, Jan.~Feb. 1947

Quantum Mechanics of Circularly Polarized Photons. II. Application to the normal Zeeman effect.

By Gentaro ARAKI.

(Received August 15, 1946)

6. Interaction of atom or molecule with photons.*

The atom or the molecule is a system which consists of electrons and a nucleus or nuclei. It is generally assumed that the nucleus consists of protons and neutrons and the wave functions of these nuclear particles satisfy the Dirac equation We denote the Dirac matrices of the n–th particle by $\rho_1^{(n)}$, $\rho_2^{(n)}$, $\rho_3^{(n)}$ and $\sigma^{(n)}$.

The interaction of this system of elementary particles (atom or molecule) with photons is given by

$$H' = \sum_n e\rho_1^{(n)}\sigma^{(n)}\boldsymbol{A}(\boldsymbol{x}_n) - \sum_{n'}(\tau^\dagger\tau)^{(n')}e\rho_1^{(n')}\sigma^{(n')}\boldsymbol{A}(\boldsymbol{x}_{n'})\,. \qquad (6\cdot1)$$

where e is an elementary positive charge, and $\boldsymbol{A}$ is the vector potential of photons given by (2·4). The former sum is to be extended over electrons, and the latter over nuclear particles. $(\tau^\dagger\tau)^{(n')}$ is a projection operator to the proton state, regarding n'–th particle.

The wave function of this system of elementary particles can be represented by a series of products of one-particle functions. We assume the Dirac equation of these one-particle functions as follows;

$$(-ic\hbar\nabla\sigma\rho_1 + mc^2\rho_3 + U - mc^2 - E)\psi = 0\,, \qquad (6\cdot2)$$

where m is the mass of the considered particle. U is a point function, and E is an energy parameter excluding a rest energy. We consider ρ_1, ρ_2, ρ_3 and σ as operators for the sake of simplicity. The eigenvalues of ρ_3 are $+1$ and -1. We denote two normalized eigenoperands of this operator corresponding to these two eigenvalues by χ^+ and χ^- respectively. The one-particle function, ψ, can be represented by a linear combination of them:

---

* Sections 1–5 are contained in Part I of this paper (Progr. Theor. Phys. 1 (1946), 125).

$$\psi=\psi^+\chi^+ +\psi^-\chi^-. \tag{6·3}$$

The equation (6·2) breaks up then into two simultaneous equations for ψ^+ and ψ^-:

$$\left.\begin{aligned} &-ic\hbar\nabla\sigma\psi^- +(U-E)\psi^+=0 \\ &-ic\hbar\nabla\sigma\psi^+ -(2mc^2+E-U)\psi^-=0 \end{aligned}\right\} \tag{6·4}$$

From the latter we have

$$\psi^-=-\frac{u}{2mc}i\hbar\sigma\nabla\psi^+ \tag{6·5}$$

where

$$u=\frac{1}{1+\dfrac{E-U}{2mc^2}}. \tag{6·6}$$

We now assume that the one-particle wave function is semi-relativistic. In this approximation, we can adopt ψ^+ as the one-particle function and put $u=1$. The matrix-elements of the interaction (6·1) are represented by a sum of matrix-elements of an interaction of one particle with photons. We can reduce the matrix-elements of this one-particle interaction using the equation (6·5). Owing to the Hermitian property of $i\sigma\nabla$ and the solenoidal property of $\boldsymbol{A}$, this reduction is carried out as follows:

$$\begin{aligned}(\psi_a, H''\psi_b)&=(\psi_a^+, e\sigma\boldsymbol{A}\psi_b^-)+(\psi_a^-, e\sigma\boldsymbol{A}\psi_b^+) \\ &=-i\frac{e\hbar}{mc}(\psi_a^+, \boldsymbol{A}\nabla\psi_b^+)+\frac{e\hbar}{2mc}(\psi_a^+, \sigma\operatorname{rot}\boldsymbol{A}\psi_b^+)\end{aligned} \tag{6·7}$$

where

$$H''=e\rho_1\sigma\boldsymbol{A}(\boldsymbol{x}) \tag{6·8}$$

is the one-particle interaction. The semi-relativistic approximation of the one-particle interaction is therefore given by

$$H''=\frac{e\hbar}{mc}(-i\boldsymbol{A}\nabla+\boldsymbol{S}\operatorname{rot}\boldsymbol{A}), \tag{6·9}$$

where $\boldsymbol{S}$ is the spin operator of the considered particle. This expression is somewhat different from the ordinarily adopted one in a non-relativistic approximation. The latter term is replaced by $\boldsymbol{A}^2c^2/(2m)$ in the ordinary expression⁽¹⁾.

The mass of the nuclear particle is very large compared with that of the electron. The latter sum of $(6\cdot1)$ is therefore neglegible compared with the former in this case. The semi-relativistic interaction of an atom or a molecule with photons is thus reduced to

$$H'=2\mu\sum_n\left\{-i\boldsymbol{A}(\boldsymbol{x}_n)\boldsymbol{r}^{(n)}+\boldsymbol{S}^{(n)}\operatorname{rot}\boldsymbol{A}(\boldsymbol{x}_n)\right\}. \qquad (6\cdot10)$$

where μ is the Bohr magneton and the summation is to be extended over all the electrons in the atom or the molecule.

From $(2\cdot4)$, $(3\cdot4)$, $(3\cdot18)$ and $(6\cdot10)$ we have

$$\left.\begin{aligned}H'&=\sum_{\text{whole } k}\left\{H_1(\boldsymbol{k})a_1(\boldsymbol{k})+H_2(\boldsymbol{k})a_2(\boldsymbol{k})\right\}+\text{its adjoint, for a linear polarization}\\&=\sum_{\text{whole } k}\left\{H_l(\boldsymbol{k})a_l(\boldsymbol{k})+H_r(\boldsymbol{k})a_r(\boldsymbol{k})\right\}+\text{its adjoint, for a circular polarization}\end{aligned}\right\} \qquad (6\cdot11)$$

where

$$H_j(\boldsymbol{k})=2i\mu\hbar c\sqrt{\frac{2\pi}{V}}\sum_n\frac{e^{i\boldsymbol{k}\boldsymbol{r}_n}}{\sqrt{\hbar\omega}}\left\{-\boldsymbol{e}_j(\boldsymbol{k})\boldsymbol{r}^{(n)}+\boldsymbol{S}^{(n)}[\boldsymbol{k}\boldsymbol{e}_j(\boldsymbol{k})]\right\} \qquad (6\cdot12)$$

$$(j=1,\ 2,\ l \text{ or } r)$$

and $\boldsymbol{e}_j$ denotes a polarization vector according to the following equations:

$$\left.\begin{aligned}&\boldsymbol{e}_j=\boldsymbol{e}_1,\ \boldsymbol{e}_2 \quad \text{for a linear polarization } (j=1,\ 2),\\&\boldsymbol{e}_l=\boldsymbol{e}_r^*=\boldsymbol{e}^+,\ \boldsymbol{e}_r=\boldsymbol{e}_l^*=\boldsymbol{e}^- \quad \text{for circular polarization } (j=l,\ r),\end{aligned}\right\} \qquad (6\cdot13)$$

$\boldsymbol{e}^\pm$ being defined by $(3\cdot17)$.

7. The normal Zeeman effect.

Consider a system which consists of an atom and photons. Let A be a state of the system in which the atom is in a state a and there is no photon, and B be another state in which the atom is in a state b and there

---

(1) W. Heitler, Quantum theory of radiation (1936).
G. Wentzel, Geiger-Scheel's Handb. der Phys., 24/1 (1933), 740.

is a $\boldsymbol{k}$-photon. The probability for the transition per unit time of the system from the state A to the state B in which $\boldsymbol{k}$ is in a differential solid angle $d\Omega$ is given by(2)

$$\frac{V\nu^2}{c^3\hbar^2}|H'_{BA}|^2 d\Omega \tag{7·1}$$

where $2\pi\nu$ is equal to ω, and $\hbar\omega$ is equal to the difference between energies of the atom in the states a and b.

From (7·1), (6·11) and (6·12) we have the energy radiated in the direction of $\boldsymbol{k}$ per unit solid angle per sec as follows:

$$\hbar\omega\frac{V\nu^2}{c^3\hbar^2}|H'_{BA}|^2=\frac{8\pi\mu^2\nu^2}{c}\left|\left(\psi_b,\ \sum e^{-i\boldsymbol{k}\boldsymbol{r}_n}\{\boldsymbol{e}\boldsymbol{r}^{(n)}+\boldsymbol{S}^{(n)}[\boldsymbol{k}\boldsymbol{e}]\}\psi_a\right)\right|^2 \tag{7·2}$$

where ψ_a and ψ_b are eigenfunctions of the atom in the states a and b respectively, and $\boldsymbol{e}$ is the polarization vector given by (6·13). The emitted photon is linearly polarized in case of $\boldsymbol{e}=\boldsymbol{e}_1(\boldsymbol{k})$ or $\boldsymbol{e}_2(\boldsymbol{k})$, right-circularly polarized in case of $\boldsymbol{e}=\boldsymbol{e}^+(\boldsymbol{k})$, and left-circularly polarized in case of $\boldsymbol{e}=\boldsymbol{e}^-(\boldsymbol{k})$.

In case of a dipole radiation, the exponential function in (7·2) can be omitted and the latter term in the brace of (7·2) can be neglected. As is well known(3), the eigenfunctions of the atom satisfy the following equation:

$$\left(\psi_b,\ \sum_n \boldsymbol{r}^{(n)}\psi_a\right)=\frac{m}{\hbar^2}(E_a-E_b)\left(\psi_b,\ \sum_n \boldsymbol{x}_n\psi_a\right) \tag{7·3}$$

where m is the mass of the electron, and E_a and E_b are energies of the atom in the states a and b respectively. The energy of light radiated from the atom in the direction of $\boldsymbol{k}$ per unit solid angle per sec is thus reduced to

$$\left(\frac{2\pi}{\nu}\right)^3\nu^4|(\psi_b,\ \boldsymbol{e}\boldsymbol{P}\psi_a)|^2 \tag{7·4}$$

where

$$\boldsymbol{P}=\sum_n e\boldsymbol{x}_n \tag{7·5}$$

---

(2) W. Pauli, Geiger-Scheel's Handbuch der Phys., 24/1 (1933), 158.

(3) H. Bethe, ibid., 429.

is the dipole moment of the atom. The selection rule is given by the condition of non vanishing of the matrix element, $(\psi_b, \boldsymbol{eP}\psi_a)$, of the transition.

We now assume that the atom is in a (strong) statical magnetic field, and that the states a and b belong to a singlet (or the levels suffer the Paschen-Back effect). In this case ν is given by$^{(4)}$

$$\nu=\nu_0-\frac{\mu}{h}H_0\varDelta M_L \qquad (7\cdot 6)$$

where ν_0 is the corresponding frequency in case of vanishing field, H_0 is the intensity of the magnetic field, and $\varDelta M_L$ is the increment of the orbital magnetic quantum number, M_L, of the atom after the transition against that before transition.

When we observe the spectrum from the positive direction of the magnetic field towards its negative direction, i. e., when $\boldsymbol{k}$ is parallel to the field, let the direction of the z–axis of the right-hand system be parallel to the field. The intensity of the right-circularly polarized light emitted from the atom is then given by $(7\cdot 4)$ substituting P_x+iP_y for $\sqrt{2}\ \boldsymbol{eP}$ in it, and that of the left-circularly polarized light is given by substituting P_x-iP_y. The matrix element$^{(5)}$ of the former operator does not vanish only when $\varDelta M_L$ is equal to $+1$, and that of the latter does not vanish only when $\varDelta M_L$ is equal to -1. In the former case the frequency of the emitted light is equal to $\nu_0-\mu H_0/h$ and in the latter case it is equal to $\nu_0+\mu H_0/h$. There is no other component, because the intensity is zero when $\varDelta M_L$ is not equal to ± 1. In other words, the component with the increased frequency is left-circularly polarized light and that with the decreased frequency is right-circularly polarized light. This is in complete agreement with the well known experimental fact.

Department of Industrial Chemistry,
Kyoto Imperial University.

---

(4) E. U. Condon and G. H. Shortley, Theory of Atomic Spectra, (1935) p. 378.

(5) Ibid.

B. L. Van der Waerden, Gruppentheoretische Methode in der Quantenmechanik, (1932) p. 74.

Progress of Theoretical Physics Vol, II, No. 1, Jan.~Feb. 1947.

On the Effect of the Field Reactions on the Interaction of Mesotrons and Nuclear Particles. III.*

By Shin-ichirô TOMONAGA.

(Received July 8, 1946)

§ 3. Treatment by Ritz's procedure.(6),(7),(8),(9)

In the foregoing two paragraphs we have dealt with the case of strong coupling. The physically interesting case will be, however, of the intermediate coupling where neither perturbation theory nor method of Wentzel converges. It is known that in such a case Ritz's procedure often affords a powerfull mean for obtaining an approximate solution. In this paragraph we shall show how this method can be applied to our problem and what are the results thus obtained.

Since the method of Ritz has been applied only to the problems concerning atoms and molecules and never to the field theory, it is desirable first to bring the latter theory into a form in which more close analogy with the former is obtained. It is thus more convenient to use the formalism in the configuration space for mesotrons, the formalism which emphasizes their particle aspect, than to use the ordinary formalism of quantized waves based on the wave picture of the mesotrons. The equivalence of these two formalisms is a well established fact and the transition from one formalism to the other is always possible.

A :—Schrödinger equation.

Our system has the Hamiltonian of the form :

$$H=\int K\{a^*(\mathfrak{k})a(\mathfrak{k})+b^*(\mathfrak{k})b(\mathfrak{k})\}d\mathfrak{k}$$
$$-\frac{l}{2}\int g(k)\Big[\{a(k)-b^*(\mathfrak{k})\}(\tau_1+i\tau_2)+\{a^*(\mathfrak{k})-b(\mathfrak{k})\}(\tau_1-i\tau_2)\Big]d\mathfrak{k}. \tag{3·1}$$

---

* §1, §2 are contained in Part I and Part II of this paper (Progr. Theor. Phys. 1 (1946), 83, 109).

(6) T. Miyazima, T. Tati and S. Tomonaga : to be published soon.

(7) S. Tomonaga : Sc. pap. I.P.C.R., 39 (1941), 247.

(8) T. Miyazima and S. Tomonaga : Sc. pap. I.P.C.R., 40 (1942), 21.

(9) S. Kanesawa, H. Jen and S. Tomonaga : Bull. I.P.C.R., in press.

What will be the Schrödinger equation for the eigen-value problem

$$(H-K_0)\Phi=0, \qquad (3\cdot 2)$$

when we use the formalism in the configuration space? In the ordinary quantum mechanics of particles the Schrödinger function Φ is a function of their coordinates $q_1, q_2, \cdots q_N$, N being their total number. Corresponding to this, we must regard Φ as a function of the coordinates of the mesotrons. In our case, however, the position coordinates can not be used, because in our case a relativistic treatment is necessary in which position of a particle is not a properly defined quantity.* We use thus the momenta $\mathfrak{k}_1$, $\mathfrak{k}_2$, $\mathfrak{k}_3, \cdots$ of the particles 1, 2, 3,...as the quantities specifying the states of the particles. Further, it must be accounted for that the number of the particles is not determined in our problem. Thus the number N of the particle itself must be regarded as an argument in the Schrödinger function Φ.

Under this circumstance we represent, following Fock,(10) the Schrödinger function Φ by a set of Schrödinger functions each corresponding to the state of no particles, of one particle, of two particles,... . In order to simplify the problem we shall first deal with the state in which the charge of the system is $+1$ (i. e. the proton state of the system or the state of $m=+1/2$ if we introduce the charge quantum number m). In such a state the set of Schrödinger functions can be arranged in the form:

$$\Phi=\left\{\begin{array}{l} \varphi_0 \\ \varphi_1(\mathfrak{k}_1^+) \\ \varphi_2(\mathfrak{k}_1^+\,;\ \mathfrak{k}_1^-) \\ \varphi_3(\mathfrak{k}_1^+,\ \mathfrak{k}_2^+\,;\ \mathfrak{k}_1^-) \\ \varphi_4(\mathfrak{k}_1^+,\ \mathfrak{k}_2^+\,;\ \mathfrak{k}_1^-,\ \mathfrak{k}_2^-) \\ \cdots\cdots\cdots\cdots \\ \cdots\cdots\cdots\cdots \end{array}\right\} \qquad (3\cdot 3)$$

The physical meaning of these functions are as follows: The absolute square $|\varphi_{2n}(\mathfrak{k}_1^+, \mathfrak{k}_2^+, \cdots \mathfrak{k}_n^+\,;\ \mathfrak{k}_1^-, \mathfrak{k}_2^-, \cdots \mathfrak{k}_n^-)|^2$ of the $2n$-th function gives the probability for that there exist n positive mesotrons with the momenta $\mathfrak{k}_1^+$, $\mathfrak{k}_2^+, \cdots \mathfrak{k}_n^+$ respectively and n negative mesotrons with the momenta $\mathfrak{k}_1^-$, $\mathfrak{k}_2^-, \cdots \mathfrak{k}_n^-$, while the nuclear particle is in the proton state. Further, the absolute square $|\varphi_{2n+1}(\mathfrak{k}_1^+, \mathfrak{k}_2^+, \dots \mathfrak{k}_{n+1}^+ ; \mathfrak{k}_1^-, \mathfrak{k}_2^-, \dots \mathfrak{k}_n^-)|^2$ of the $2n+1$–th function gives the probability for that there exist $n+1$ positive mesotrons with the momenta $\mathfrak{k}_1^+, \mathfrak{k}_2^+, \dots \mathfrak{k}_{n+1}^+$ respectively and n mesotrons with the momenta $\mathfrak{k}_1^-, \mathfrak{k}_2^-, \dots \mathfrak{k}_n^-$ respectively while the nuclear particle is in the neutron state. As stated above, our function (3 · 3) represents the state with $m = 1/2$. This restriction will be removed later.

---

* See, for instance, Pauli's article in Handbuch d. Physik XIV/1, 260.

(10) V. Fock: Z. phys., 75 (1932), 622.

When the function Φ is put in the form of (3 · 3), the Schrödinger equation (3 · 2) is a system of integral equations for $\varphi_0, \psi_1, \varphi_2, \dots$. According to Fock, this system of equations is expressed in the form:

$$\left.\begin{aligned}
&K_0\varphi_0 = -l\int \psi_1(\mathfrak{k}_1^+)\,g(k_1^+)\,d\mathfrak{k}_1^+ \\
&(K_0 - K_1^+)\psi_1(\mathfrak{k}_1^+) = -l\Big\{g(k_1^+)\varphi_0 - \int \varphi_2(\mathfrak{k}_1^+ ; \mathfrak{k}_1^-)\,g(k_1^-)\,d\mathfrak{k}_1^-\Big\} \\
&(K_0 - K_1^+ - K_1^-)\varphi_2(\mathfrak{k}_1^+ ; \mathfrak{k}_1^-) = l\Big\{g(k_1^-)\,\psi(\mathfrak{k}_1^+) - \sqrt{2}\int \psi_3(\mathfrak{k}_1^+, \mathfrak{k}_2^+ ; \mathfrak{k}_1^-)\,g(k_2^+)\,d\mathfrak{k}_2^+\Big\} \\
&(K_0 - K_1^+ - K_1^- - K_2^+)\psi_3(\mathfrak{k}_1^+, \mathfrak{k}_2^+ ; \mathfrak{k}_1^-) \\
&\qquad = -l\Big\{\frac{1}{\sqrt{2}}\sum_2^{+} g(k_2^+)\,\varphi_2(\mathfrak{k}_1^+ ; \mathfrak{k}_1^-) - \sqrt{2}\int \varphi_4(\mathfrak{k}_1^+, \mathfrak{k}_2^+ ; \mathfrak{k}_1^-, \mathfrak{k}_2^-)\,g(k_2^-)\,d\mathfrak{k}_2^-\Big\} \\
&(K_0 - K_1^+ - K_1^- - K_2^+ - K_2^-)\varphi_4(\mathfrak{k}_1^+, \mathfrak{k}_2^+ ; \mathfrak{k}_1^-, \mathfrak{k}_2^-) \\
&\qquad = l\Big\{\frac{1}{\sqrt{2}}\sum_2^{-} g(k_2^-)\,\psi_3(\mathfrak{k}_1^+, \mathfrak{k}_2^+ ; \mathfrak{k}_1^-) - \sqrt{3}\int \psi_5(\mathfrak{k}_1^+, \mathfrak{k}_2^+, \mathfrak{k}_3^+ ; \mathfrak{k}_1^-, \mathfrak{k}_2^-)\,g(k_3^+)\,d\mathfrak{k}_3^+\Big\} \\
&\qquad\qquad\qquad\qquad \cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
&\qquad\qquad\qquad\qquad \cdots\cdots\cdots\cdots\cdots\cdots\cdots
\end{aligned}\right\} \tag{3 · 4}$$

The symbol $\sum\limits_n^{+}$ in (3 · 4) means the operation of summing up the n functions obtained by interchanging cyclically the n variables for the positive particles in the function standing behind this symbol, thus for instance:

$$\begin{aligned}
\sum_3^{+} f(\mathfrak{k}_1^+, \mathfrak{k}_2^+, \mathfrak{k}_3^+ ; \mathfrak{k}_1^-, \mathfrak{k}_2^-, \dots) &= f(\mathfrak{k}_1^+, \mathfrak{k}_2^+, \mathfrak{k}_3^+ ; \mathfrak{k}_1^-, \mathfrak{k}_2^-, \dots) \\
&\quad + f(\mathfrak{k}_2^+, \mathfrak{k}_3^+, \mathfrak{k}_1^+ ; \mathfrak{k}_1^-, \mathfrak{k}_2^-, \dots) + f(\mathfrak{k}_3^+, \mathfrak{k}_1^+, \mathfrak{k}_2^+ ; \mathfrak{k}_1^-, \mathfrak{k}_2^-, \dots).
\end{aligned}$$

The symbol $\sum\limits_n^{-}$ is the corresponding operation on the variables for the negative particles.

B :—Hartree approximation.

When we set up the equation in the form of (3 · 4), we are suggested to solve our problem by means of Ritz's procedure when neither the expantion in poweres of l nor l^{-1} converges. Our problem is, namely, a kind of many body problem which will be solved by means of the Hartree method.

By Hartree approximation is meant that one obtains an approximate solution by putting the wave functions $\varphi(\mathfrak{k}_1^+, \mathfrak{k}_2^+, \dots ; \mathfrak{k}_1^-, \mathfrak{k}_2^-, \dots)$ and $\psi(\mathfrak{k}_1^+, \mathfrak{k}_2^+, \dots ; \mathfrak{k}_1^-, \mathfrak{k}_2^-, \dots)$ in products of functions each containing only one variable :

$$\left.\begin{aligned} \varphi(\mathfrak{k}_1^+, \mathfrak{k}_2^+, \dots ; \mathfrak{k}_1^-, \mathfrak{k}_2^-, \dots) &= \text{const}\, f_+(\mathfrak{k}_1^+) f_+(\mathfrak{k}_2^+) \dots f_-(\mathfrak{k}_1^-) f_-(\mathfrak{k}_2^-) \dots \\ \psi(\mathfrak{k}_1^+, \mathfrak{k}_2^+, \dots ; \mathfrak{k}_1^-, \mathfrak{k}_2^-, \dots) &= \text{const}\, f_+(\mathfrak{k}_1^+) f_+(\mathfrak{k}_2^+) \dots f_-(\mathfrak{k}_1^-) f_-(\mathfrak{k}_2^-) \dots , \end{aligned}\right\} \quad (3 \cdot 5)$$

and determining the factors $f_+(\mathfrak{k})$, $f_-(\mathfrak{k})$ and the constants involved in (3 · 5) so that the product functions satisfy the equation (3 · 4) as approximately as possible.

The problem is, however, still too complicated if one will find the most approximate $f_+(\mathfrak{k})$ and $f_-(\mathfrak{k})$ among all possible functions. So, we will simplify the problem in such a way that we look for the most approximate $f_+(\mathfrak{k})$ and $f_-(\mathfrak{k})$ among functions of the form

$$\left.\begin{aligned} f_+(\mathfrak{k}) &= \frac{g(k)}{K-\alpha} \\ f_-(\mathfrak{k}) &= \frac{g(k)}{K-\beta} , \end{aligned}\right\} \quad (3 \cdot 6)$$

where α and β are considered as variation parameters. Whether the true wave functions can be approximated well by this form can only be decided by results, but that this will give a good approximation can be expected by the following fact :

Suppose an imaginary problem in which only neutral longitudinal mesotrons interact with the nuclear particle. The Hamiltonian function for this problem is then

$$H = \int K c^*(\mathfrak{k}) c(\mathfrak{k})\, d\mathfrak{k} - \frac{l}{\sqrt{2}} \int g(k) \left\{ c(\mathfrak{k}) + c^*(\mathfrak{k}) \right\} d\mathfrak{k}. \qquad (3 \cdot 7)$$

Since this Hamiltonian has the form of (2 · 31), it is obvious that this problem is accurately solved by means of the displacement of origin for the amplitudes c and c^* :

$$\bar{c}(\mathfrak{k})=c(\mathfrak{k})-\frac{l}{\sqrt{2}}\frac{g(k)}{K},\quad \bar{c}^*(\mathfrak{k})=\bar{c}^*(\mathfrak{k})-\frac{l}{\sqrt{2}}\frac{g(k)}{K} \qquad (3\cdot 8)$$

Now, suppose that the Schrödinger equation $(H-K_0)\Phi=0$ for this problem is written down according to the formalism in the configuration space. In this formalism the probability amplitude Φ is represented by a set of functions χ_0, $\chi_1(\mathfrak{k}_1)$, $\chi_2(\mathfrak{k}_1, \mathfrak{k}_2)$,... and the Schrödinger equation is the system of integral equations of the form similar with (3·4):

$$\left.\begin{aligned}
&K_0\chi_0=-\frac{l}{\sqrt{2}}\int\chi(\mathfrak{k}_1)g(k_1)d\mathfrak{k}_1\\
&(K_0-K_1)\chi_1(\mathfrak{k}_1)=-\frac{l}{\sqrt{2}}\Big\{g(k_1)\chi_0+\sqrt{2}\int\chi_2(\mathfrak{k}_1,\ \mathfrak{k}_2)g(k_2)d\mathfrak{k}_2\\
&(K_0-K_1-K_2)\chi_2(\mathfrak{k}_1,\ \mathfrak{k}_2)\\
&\qquad=-\frac{l}{\sqrt{2}}\Big\{\frac{1}{\sqrt{2}}\sum_2 g(k_2)\chi_1(\mathfrak{k}_1)+\sqrt{3}\int\chi_3(\mathfrak{k}_1,\ \mathfrak{k}_2,\ \mathfrak{k}_3)g(k_3)d\mathfrak{k}_3\\
&(K_0-K_1-K_2-K_3)\chi_3(\mathfrak{k}_1,\ \mathfrak{k}_2,\ \mathfrak{k}_3)\\
&\quad=-\frac{l}{\sqrt{2}}\Big\{\frac{1}{\sqrt{3}}\sum_3 g(k_3)\chi_2(\mathfrak{k}_1,\ \mathfrak{k}_2)+\sqrt{4}\int\chi_4(\mathfrak{k}_1,\ \mathfrak{k}_2,\ \mathfrak{k}_3,\ \mathfrak{k}_4)g(k_4)d\mathfrak{k}_4\\
&\quad\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
&\quad\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\ .
\end{aligned}\right\} \qquad (3\cdot 9)$$

This simultaneous integral equations must be solved accurately because the problem can be solved, as mentioned above, by means of the method of quantized amplitudes. In fact, it can be verified easily that

$$\left.\begin{aligned}
\chi_n(\mathfrak{k}_1,\ \mathfrak{k}_2,\ldots,\ \mathfrak{k}_n)&=\left(\frac{l}{\sqrt{2}}\right)^n\frac{1}{\sqrt{n!}}\,\frac{g(k_1)}{K_1}\cdot\frac{g(k_2)}{K_2}\cdot\ldots\frac{g(k_n)}{K_n}\\
K_0&=-\frac{l^2}{2}\int\frac{g(k)^2}{K}d\mathfrak{k}
\end{aligned}\right\} \qquad (3\cdot 10)$$

satisfy (3·9) accurately.

One sees that in this problem the solution has really the form of (3·5), the factor being given by

$$f(\mathfrak{k})=\frac{g(k)}{K}\,.$$

The results (1·22) obtained classically in § 1 show, moreover, that in case of charged mesotron the function $g(k)/(K-\omega)$ and $g(k)/(K+\omega)$ appear respectively for positive and negative mesotrons in the place where the function $g(k)/K$ appears for the neutral mesotrons. It seems thus natural to assume the functions of the form (3·6) in our problem of charged mesotrons.

In this way our problem is to determine the unknown parameters $c_{0,0}$, $c_{1,0}$, $c_{1,1}, \ldots$ and α and β so that the set of Schrödinger functions

$$\left.\begin{aligned}\varphi_{2n}(\mathfrak{k}_1^+,\ \mathfrak{k}_2^+, \ldots \mathfrak{k}_n^+\ ;\ \mathfrak{k}_1^-,\ \mathfrak{k}_2^-, \ldots \mathfrak{k}_n^-) &= (-1)^n \frac{c_{n,n}}{K_2^n L_2^n} \prod_{\lambda=1}^{n} \frac{g(k_\lambda^+)}{K_\lambda^+ - \alpha} \prod_{\nu=1}^{n} \frac{g(k_\nu^-)}{K_\nu^- - \beta} \\ \psi_{2n+1}(\mathfrak{k}_1^+,\ \mathfrak{k}_2^+, \ldots \mathfrak{k}_{n+1}^+\ ;\ \mathfrak{k}_1^-,\ \mathfrak{k}_2^-, \ldots \mathfrak{k}_n^-) &= (-1)^n \frac{c_{n+1,n}}{K_2^{n+1} L_2^n} \prod_{\lambda=1}^{n+1} \frac{g(k_\lambda^+)}{K_\lambda^+ - \alpha} \prod_{\nu=1}^{n} \frac{g(k_\nu^-)}{K_\nu^{-1} - \beta}\end{aligned}\right\} \tag{3·11}$$

satisfy the equation (3·4) as accurately as possible. We have introduced here the abreviations

$$K_2 = \sqrt{\int \frac{g(k)^2}{(K-\alpha)^2}\, d\mathfrak{k}}, \quad L_2 = \sqrt{\int \frac{g(k)^2}{(K-\beta)^2}\, d\mathfrak{k}}. \tag{3·12}$$

We shall use often the abreviations

$$K_n = \left\{\int \frac{g(k)^2}{(K-\alpha)^n}\, d\mathfrak{k}\right\}^{1/(4-n)} \qquad L_n = \left\{\int \frac{g(k)^2}{(K-\beta)^n}\, d\mathfrak{k}\right\}^{1/(4-n)} \tag{3·13}$$

which have each the dimention of energy (or of 1/length).

As is well known, we have to determine c, α and β so as to make the expectation value of H formed by the wave function under consideration as small as possible. If this expectation value is denoted by $\overline{K}(c_{0,0},\ c_{1,0},\ c_{1,1}, \ldots\ ;\ \alpha,\ \beta)$, this is given by

$$\left.\begin{aligned}&\overline{K}(c_{0,0},\ c_{1,0},\ c_{1,1}, \ldots\ ;\ \alpha,\ \beta) \\ &= \sum_{n=0}^{\infty} \left\{ n\left(\frac{K_1^3}{K_2^2} + \alpha\right) + n\left(\frac{L_1^3}{L_2^2} + \beta\right) \right\} c_{n,n}^2 + \sum_{n=0}^{\infty} \left\{ (n+1)\left(\frac{K_1^3}{K_2^2} + \alpha\right) + n\left(\frac{L_1^3}{L_2^2} + \beta\right) \right\} c_{n+1,n}^2 \\ &\qquad - \frac{2lK_1^3}{K_2} \sum_{n=0}^{\infty} \sqrt{n+1}\, c_{n,n}\, c_{n+1,n} - \frac{2lL_1^3}{L_2} \sum_{n=0}^{\infty} \sqrt{n+1}\, c_{n+1,n}\, c_{n+1,n+1}, \\ &\text{where} \qquad \sum_{n=0}^{\infty} c_{n,n}^2 + \sum_{n=0}^{\infty} c_{n+1,n}^2 = 1\end{aligned}\right\} \tag{3·14}$$

In determining the values of c, a and β which make $\overline{K}$ minimum, we first consider that α and β have some fixed value. Thus the minimum value of $\overline{K}$ is obtained for these specified values of α and β. We determine then α and β so that this $\overline{K}$ has the smallest value. As $\overline{K}(c_{0,0}, c_{1,0}, c_{1,1}, \dots ; \alpha, \beta)$ is a quadratic form of c's, our variation problem gives rise to the following system of linear equations for $c_{0,0}, c_{1,0}, c_{1,1}, \dots$:

$$\left.\begin{aligned} &-\frac{lL_1^3}{L_2}\sqrt{n}\, c_{n,n+1}+\left\{n\left(\frac{K_1^3}{K_2^2}+\alpha\right)+n\left(\frac{L_1^3}{L_2^2}+\beta\right)-E\right\}c_{n,n} \\ &\qquad\qquad -\frac{lK_1^3}{K_2}\sqrt{n+1}\, c_{n+1,n}=0 \\ &-\frac{lK_1^3}{K_2}\sqrt{n+1}\, c_{n,n}+\left\{(n+1)\left(\frac{K_1^3}{K_2^2}+\alpha\right)+n\left(\frac{L_1^3}{L_2^2}+\beta\right)-E\right\}c_{n+1,n} \\ &\qquad\qquad -\frac{lL_1^3}{L_2}\sqrt{n+1}\, c_{n+1,n+1}=0 \\ &\qquad\qquad n=0,\ 1,\ 2\dots . \end{aligned}\right\} \tag{3・15}$$

The eigen-value E of this system of equations gives the minimum value of $\overline{K}$ for the specified values of α and β.

It is often more convenient to transform our system of equations into a system of differectial equations. For this purpose we introduce the functions $\varphi(N_+, N_-)$ and $\psi(N_+, N_-)$ of two variables N_+ and N_- having the following properties: the function $\varphi(N_+, N_-)$ has the values $c_{n,n}$ at the point $N_+=n$, $N_-=n$, and the function $\psi(N_+, N_-)$ the value $c_{n+1,n}$ at the point $N_+=n+1$, $N_-=n$. Thus

$$\left.\begin{aligned} \varphi(N_+, N_-)_{\substack{N_+=n\\ N_-=n}}&=\varphi(n, n)=c_{n,n} \\ \psi(N_+, N_-)_{\substack{N_+=n+1\\ N_-=n}}&=\psi(n+1, n)=c_{n+1,n} . \end{aligned}\right\} \tag{3・16}$$

We introduce further the operators A_+ and A_+^* which, operated on an arbitrary function $F(N_+, N_-)$ of N_+ and N_-, generate the new functions $\sqrt{N_++1}\,F(N_++1, N_-)$ and $\sqrt{N_+}\,F(N_+-1, N_-)$ respectively:

$$\left.\begin{aligned} A_+F(N_+, N_-)&=\sqrt{N_++1}\,F(N_++1, N_-) \\ A_+^*F(N_+, N_-)&=\sqrt{N_+}\,F(N_+-1, N_-) . \end{aligned}\right\} \tag{3・17$_+$}$$

Further let A_- and A_-^* be the corresponding operators having the properties:

$$\left.\begin{aligned} A_- F(N_+,\ N_-)&=\sqrt{N_-+1}\,F(N_+,\ N_-+1)\\ A_-^* F(N_+,\ N_-)&=\sqrt{N_-}\,F(N_+,\ N_--1)\,. \end{aligned}\right\} \qquad (3\cdot 17)$$

These operators satisfy the following relations:

$$\left.\begin{gathered} [A_+,\ A_+^*]=[A_-,\ A_-^*]=1\\ [A_+,\ A_-]=[A_+^*,\ A_-^*]=[A_+,\ A_-^*]=[A_+^*,\ A_-]=0\\ A_+^*A_+=N_+,\ A_-^*A_-=N_-\,. \end{gathered}\right\} \qquad (3\cdot 18)$$

We introduce next the two-row function Φ whose first row is $\varphi(N_+,\ N_-)$ defined above and the second row is $\varphi(N_+,\ N_-)$ of $(3\cdot 16)$:

$$\Phi(N_+,\ N_-)=\begin{pmatrix}\varphi(N_+,\ N_-)\\ \psi(N_+,\ N_-)\end{pmatrix} \qquad (3\cdot 19)$$

Further let τ_1, τ_2 and τ_3 be the matrices

$$\tau_1=\begin{pmatrix}0&1\\1&0\end{pmatrix},\quad \tau_2=\begin{pmatrix}0&-i\\i&0\end{pmatrix},\quad \tau_3=\begin{pmatrix}1&0\\0&-1\end{pmatrix}. \qquad (3\cdot 20)$$

Now using these operators and matrices the equation $(3\cdot 15)$ can be written in the compact form:

$$\begin{aligned} \Big[\Big\{&\Big(\frac{K_1^3}{K_2^2}+\alpha\Big)A_+^*A_++\Big(\frac{L_1^3}{L_2^2}+\beta\Big)A_-^*A_-\Big\}\\ &-\frac{l}{2}\Big\{\Big(\frac{K_1^3}{K_2}A_++\frac{L_1^3}{L_2}A_-^*\Big)(\tau_1+i\tau_2)\\ &+\Big(\frac{K_1^3}{K_2}A_+^*+\frac{L_1^3}{L_1}A_-\Big)(\tau_1-i\tau_2)\Big\}-E\Big]\Phi=0\,. \end{aligned} \qquad (3\cdot 21)$$

In order to transform $(3\cdot 21)$ into a system of differential equations, we use the transformation theory of the quantum mechanics: We introduce first the operators ξ_1, ξ_2 and p_1, p_2 by means of

$$\left.\begin{aligned} A_++A_-^*&=\xi_1-i\xi_2\,, & A_+-A_-^*&=p_2+ip_1\\ A_+^*+A_-&=\xi_1+i\xi_2\,, & A_+^*-A_-&=p_2-ip_1 \end{aligned}\right\}, \qquad (3\cdot 22)$$

then these ξ_1, ξ_2 and p_1, p_2 are all hermitic and satisfy the commutation relations

$$[\xi_j,\ p_{j'}]=i\delta_{jj'}\,,\quad [\xi_j,\ \xi_{j'}]=[p_j,\ p_{j'}]=0\,. \qquad (3\cdot 23)$$

Thus, if we use ξ-representation, the function Φ is represented by a two-row function of ξ_1 and ξ_2. The operators ξ_1 and ξ_2 are in this representation the multiplication of this function by ξ_1 and ξ_2, respectively and the operators p_1 and p_2 are respectively the differentiation $1/i \cdot \partial/\partial\xi_1$ and $1/i \cdot \partial/\partial\xi_2$ applied to this function. Thus in this representation (3·21) is represented by the following system of differential equations:

$$\left.\begin{aligned}
&\Big\{-\frac{1}{2}\Big(\frac{\partial^2}{\partial\xi_1^2}+\frac{\partial^2}{\partial\xi_2^2}\Big)+\frac{1}{2}(\xi_1^2+\xi_2^2)+ia\Big(\xi_1\frac{\partial}{\partial\xi_2}-\xi_2\frac{\partial}{\partial\xi_1}\Big)-1-W'_0\Big\}\varphi(\xi_1,\ \xi_2)\\
&\qquad\qquad-\Big\{v(\xi_1+i\xi_2)+w\Big(\frac{\partial}{\partial\xi_1}+i\frac{\partial}{\partial\xi_2}\Big)\Big\}\psi(\xi_1,\ \xi_2)=0\\
&\Big\{-\frac{1}{2}\Big(\frac{\partial^2}{\partial\xi_1^2}+\frac{\partial^2}{\partial\xi_2^2}\Big)+\frac{1}{2}(\xi_1^2+\xi_2^2)+ia\Big(\xi_1\frac{\partial}{\partial\xi_2}-\xi_2\frac{\partial}{\partial\xi_1}\Big)-1-W_0\Big\}\psi(\xi_1,\ \xi_2)\\
&\qquad\qquad-\Big\{v(\xi_1-i\xi_2)-w\Big(\frac{\partial}{\partial\xi_1}-i\frac{\partial}{\partial\xi_2}\Big)\Big\}\varphi(\xi_1,\ \xi_2)=0\,.
\end{aligned}\right| \tag{3·24}$$

where a, v, w and W'_0 are the following abreviations:

$$\left.\begin{aligned}
a&=\Big(\frac{K_1^3}{K_2^2}+\alpha+\frac{L_1^3}{L_2^2}+\beta\Big)^{-1}\Big(\frac{K_1^3}{K_2^2}+\alpha-\frac{L_1^3}{L_2^2}-\beta\Big)\\
v&=l\Big(\frac{K_1^3}{K_2^2}+\alpha+\frac{L_1^3}{L_2^2}+\beta\Big)^{-1}\Big(\frac{K_1^3}{K_2}+\frac{L_1^3}{L_2}\Big)\\
w&=l\Big(\frac{K_1^3}{K_2^2}+\alpha+\frac{L_1^3}{L_2^2}+\beta\Big)^{-1}\Big(\frac{K_1^3}{K_2}-\frac{L_1^3}{L_2}\Big)\\
W_0&=2\Big(\frac{K_1^3}{K_2^2}\,4\,\alpha+\frac{L_1^3}{L_2^2}+\beta\Big)^{-1}E
\end{aligned}\right| \tag{3·25}$$

If we introduce the polar coordinates r and θ by means of

$$\left.\begin{aligned}
\xi_1&=r\cos\theta\\
\xi_2&=r\sin\theta\,,
\end{aligned}\right\} \tag{3·26}$$

it is immediately seen that (3·24) has the solution of the form

$$\begin{pmatrix}\varphi(r,\ \theta)\\ \psi(r,\ \theta)\end{pmatrix}=\frac{1}{\sqrt{r}}\begin{pmatrix}f(r)e^{i(m-\frac{1}{2})\theta}\\ g(r)e^{i(m+\frac{1}{2})\theta}\end{pmatrix}, \tag{3·27}$$

$$m=\cdots-\frac{3}{2}\,,\ -\frac{1}{2}\,,\ \frac{1}{2}\,,\ \frac{3}{2}\,,\cdots\cdots$$

where $f(r)$ and $g(r)$ satisfy

$$\left.\begin{aligned}&\left\{-\frac{d^2}{dr^2}+\frac{m(m-1)}{r^2}+r^2+2a\left(m-\frac{1}{2}\right)-2(W_0+1)\right\}f(r)\\&\qquad-2wrg(r)-2w\left(\frac{d}{dr}+\frac{m}{r}\right)g(r)=0\\&\left\{-\frac{d^2}{dr^2}+\frac{m(m+1)}{r^2}+r^2+2a\left(m+\frac{1}{2}\right)-2(W_0+1)\right\}g(r)\\&\qquad-2wrf(r)+2w\left(\frac{d}{dr}-\frac{m}{r}\right)f(r)=0\end{aligned}\right\} \tag{3·28}$$

Our problem has been thus reduced to the eigen-value problem for the system of differential equations (3·28). But it is to be noted that the equation (3·24) is more comprehensive than the original equation (3·15): the formar equation has more number of solutions than the latter. It can be easily shown that among many solutions of (3·24) that with $m=1/2$ corresponds just to the solution of the original equation (3·15). But it can be also shown that the solutions of (3·24) with other values of m are by no means superflous: each of these solutions corresponds to the state of our system with the charge number m. In this way our restriction introduced first saying that the state under consideration is of the charge $+1$ is automatically removed if we treat the equation (3·28) generally, allowing for m also other values than 1/2.

C :—Two limiting cases;

In order to see whether our Hartree approximation will really give reliable results, we shall compare our results with those obtained by the perturbation theory on one hand and by the method of Wentzel on the other hand in two limiting cases of small and large l.

(i) Case of small l.

Since it is known from the beginning that in this case no proton-isobars exist, we consider only the state of $m=1/2$. In this case of small l it is simpler to deal with the original equation (3·15) than with the differential equation (3·28)

When we take $c_{0,0}=1$, then it is seen that $c_{1,0}=0(l)$, $c_{1,1}=0(l^2)$.... So it suffices that only $c_{0,0}$ and $c_{1,0}$ are taken into account, the higher coefficients being all neglected. In this way the approximation correct up to the order of l^2 will be obtained. The equation (3 · 15) is then reduced to

$$\left.\begin{aligned}-Ec_{0,0}-\frac{lK_1^3}{K_2}c_{1,0}=0\\-\frac{lK_1^3}{K_2}c_{0,0}+\left\{\left(\frac{K_1^6}{K_2^2}+a\right)-E\right\}c_{1,0}=0\end{aligned}\right\}\tag{3 · 29}$$

The eigen-value E is now obtained by solving the secular equation of (3 · 29):

$$E=-\frac{l^2K_1^6}{K_1^3+K_2^2a}.\tag{3 · 39}$$

this gives the eigen-value correct up to the order of l^2

We now determine a so as to make this E minimum. Then we obtain

$$\left.\begin{aligned}a=0\\E_{\min}=-l^2\boldsymbol{K}_1^3\end{aligned}\right\}\tag{3 · 31}$$

and hence

$$\left.\begin{aligned}c_{0,0}=1,\quad c_{1,0}=l\boldsymbol{K}_2\\\Phi=\begin{Bmatrix}1\\l\,\dfrac{g(k_1^+)}{K_1^+}\\\vdots\end{Bmatrix}\end{aligned}\right\}\tag{3 · 32}$$

is the required approximate solution of our problem.

The eigen-value E in (3 · 31) and the eigen-function Φ of (3 · 32) agree perfectly with the results of the perturbation theory. We see, thus, that our Hartree method applied to the case of weak coupling gives the sufficient approximation [as to the treatment by means of the perturbation theory see (3 · 52)].

(ii) Case of large l.

In this case the dimensionless quantity v contained in (3 · 28) is very large, of the order of $l\boldsymbol{K}$. This fact corresponds to the circumstance that in (2 · 15) the quantity $\sqrt{\boldsymbol{K}_{-1}^5}\,l$ was very large. Indeed, the equations (3 · 28) and (2 · 15) have substantially the same form, so that (3 · 28) can be solved in the same way as (2 · 15), using the analogy with the theory of diatomic molecules.

In this way we find that in the first approximation, the solution is of the form:

$$f(r)=g(r) \tag{3·33}$$

with $f(r)$ satisfying

$$\left\{-\frac{d^2}{dr^2}+\frac{m^2}{r^2}+r^2+2\alpha m-2(W_0+1)-2vr-\frac{2wm}{r}\right\}f(r)=0\,. \tag{3·33$'$}$$

The eigen-values and the eigen-functions for the lowest states are then

$$\left.\begin{aligned} W_0&=-\frac{1}{2}v^2-\frac{1}{2}+\frac{m^2}{2v^2}-\left\{\frac{w}{v}-\alpha\right\}m \\ f(r)&=e^{-\frac{1}{2}(r-v)^2}\,. \end{aligned}\right\} \tag{3·34}$$

We obtain then the energy $E(\alpha,\beta)$ using (3·25). This calculation can be simplified when we first expand the quantities K and L in (3·25) in terms of powers of α and β. This procedure is allowed, as our result will show that these α and β are small:

$$\left.\begin{aligned} K_1^3&=\boldsymbol{K}_1^3+\boldsymbol{K}_2^2\alpha+\boldsymbol{K}_3\alpha^2+\cdots \\ K_2^2&=\boldsymbol{K}_2^2+2\boldsymbol{K}_3\alpha^2+\cdots \\ L_1^3&=\boldsymbol{K}_1^3+\boldsymbol{K}_2^2\beta+\boldsymbol{K}_3\beta^2+\cdots \\ L_2^2&=\boldsymbol{K}_2^2+2\boldsymbol{K}_3\beta^2+\cdots \end{aligned}\right\} \tag{3·35}$$

We obtain thus

$$\begin{aligned} E(\alpha,\beta)=&-\frac{1}{2}l^2\boldsymbol{K}_1^3-\frac{1}{2}\frac{\boldsymbol{K}_1^3}{\boldsymbol{K}_2}+\frac{1}{2l^2}\frac{\boldsymbol{K}_1^3}{\boldsymbol{K}_2^4}m^2 \\ &+\left(\frac{\boldsymbol{K}_3}{\boldsymbol{K}_2^2}-\frac{\boldsymbol{K}_2^2}{\boldsymbol{K}_1^3}\right)\left[\frac{\boldsymbol{K}_1^3}{\boldsymbol{K}_2^2}\frac{\alpha+\beta}{2}-\frac{\boldsymbol{K}_1^3}{\boldsymbol{K}_2^2}m\frac{\alpha-\beta}{2}+\frac{l^2}{4}\frac{\boldsymbol{K}_1^3\boldsymbol{K}_3}{\boldsymbol{K}_2^2}(\alpha^2+\beta^2)\right. \\ &\left.-\frac{l^2}{8}\left(\frac{\boldsymbol{K}_3}{\boldsymbol{K}_2^2}-\frac{\boldsymbol{K}_2^2}{\boldsymbol{K}_1^3}\right)\boldsymbol{K}_1^3(\alpha+\beta)^2\right]. \end{aligned} \tag{3·36}$$

Now we have to determine α and β so that (3·36) becomes minimum. This gives then

$$\left.\begin{aligned} \alpha&=\frac{1}{l^2\boldsymbol{K}_3}\left(m-\frac{\boldsymbol{K}_1^3\boldsymbol{K}_3}{\boldsymbol{K}_2^4}\right) \\ \beta&=\frac{1}{l^2\boldsymbol{K}_3}\left(-m-\frac{\boldsymbol{K}_1^3\boldsymbol{K}_3}{\boldsymbol{K}_2^4}\right) \end{aligned}\right\} \tag{3·37}$$

The corresponding minimum value of E is

$$E_{\min}=-\frac{1}{2}l^2\boldsymbol{K}_1^{\,3}-\frac{1}{2}\left\{\frac{\boldsymbol{K}_1^{\,3}}{\boldsymbol{K}_2^{\,2}}+\frac{1}{l^2}\left(\frac{\boldsymbol{K}_3}{\boldsymbol{K}_2^{\,2}}-\frac{\boldsymbol{K}_2^{\,2}}{\boldsymbol{K}_2^{\,3}}\right)\frac{\boldsymbol{K}_1^{\,6}}{\boldsymbol{K}_2^{\,6}}\right\}+\frac{m^2}{2l^2\boldsymbol{K}_3} \qquad (3\cdot 38)$$

This E is to be interpreted as the self-energy of the nuclear particle in our Hartree approximation.

If one compares $(3\cdot 38)$ with $(1\cdot 33)$ and $(2\cdot 45)$, one sees that the first and the last terms of $(3\cdot 38)$ agree perfectly with two terms in $(1\cdot 33)$ or $(2\cdot 45)$. The second term of $(3\cdot 38)$ represents the quantum-theoretical zero-point energy in the Hartree approximation. The corresponding term was omitted in $(2\cdot 45)$.

Not only our Hartree approximation gives the self-energy which agrees with that obtained by Wentzel's method, our method gives also the expectation value for the self-field agreeing with Wentzel's approximation: We expand the functions $f_+(\mathfrak{k})=g(k)/K-\alpha$ and $f_-(\mathfrak{k})=g(k)/K-\beta$ appearing in $(3\cdot 11)$ in powers of α and β using $(3\cdot 27)$:

$$f_\pm(\mathfrak{k})=\frac{g(k)}{K}\pm\frac{g(k)}{K^2}\,\frac{m}{l\boldsymbol{K}_3}+\frac{g(k)}{K^2}\,\frac{\boldsymbol{K}_1^{\,3}}{l^2\boldsymbol{K}_2^{\,4}}\,. \qquad (3\cdot 39)$$

It can be then shown that the first term in $(3\cdot 39)$ corresponds to the shift of equilibrium positions $l\left[g(k)/K\sqrt{K}\right]$ in $(2\cdot 33)$, and the second term to the shift of equilibrium position $(m/l\boldsymbol{K}_3)\,Y_s$ in $(2\cdot 37)$.

Further, we see, comparing $(3\cdot 37)$ with $(1\cdot 33)$, that the m–proportional terms in the expression for α and β correspond to the angular velocity ω of the precession in the classical theory. The other terms in $(3\cdot 37)$ which are independent of m are not contained in the classical formula for ω. Analogous m–independent terms will be obtained also by Wentzel's treatment from the term R in $(2\cdot 40)$. Thus these m–independent terms in $(3\cdot 37)$ represent a quantum-theoretical effect. The physical consequence caused by the existence of these terms will be mentioned afterwards (in E of this paragraph and in B of the next paragraph).

D:—Case of intermediate coupling.

We have seen that in two limiting cases of weak and strong couplings the Hartree approximation gives the results which agree sufficiently good with the results of perturbation theory on one hand and Wentzel's method on the other hand. We may expect, therefore, that also in case of intermediate coupling our method will give sufficiently satisfactory results.

When the coupling is neither weak nor strong, we must solve (3 · 15) or (3 · 28) numerically. Since this calculation is rather tedious, we have carried out this simplifying the calculation by putting $\alpha=\beta=0$. Moreover we have restricted ourselves to the states of $|m|=1/2$. Owing to the former simplification the results obtained are less accurate, but they will be still accurate enough to give the general feature of the behavior of our system.

When we put $\alpha=\beta=0$, the system of equations (3 · 15) becomes very simple. Introducing two dimensionless quantities

$$V=l\boldsymbol{K}_2 \tag{3 · 40}$$

and

$$W=E\frac{\boldsymbol{K}_2^2}{\boldsymbol{K}_1^3}\,. \tag{3 · 41}$$

(3 · 15) reduces to

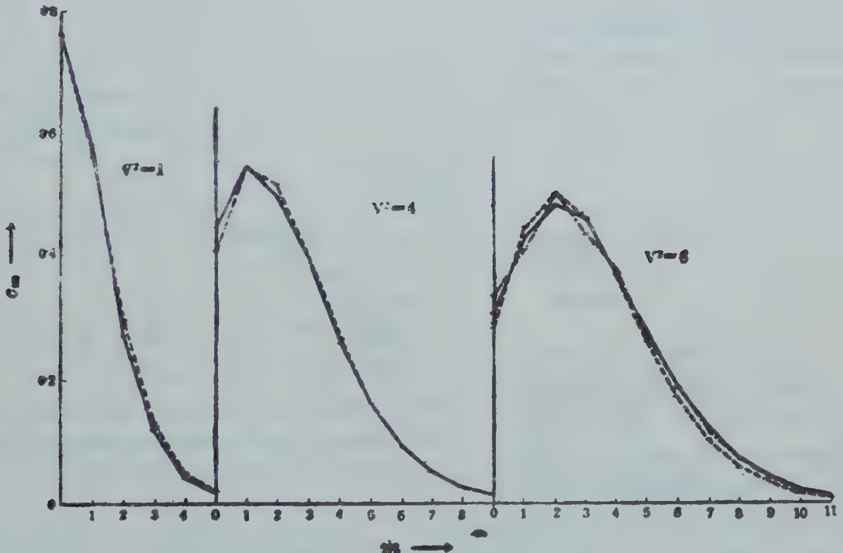


Fig. 3. Eigen-solution c_m for various values of V. Full lines: c_m obtained by the numerical calculation. Broken lines: c_m given by the approximate formula (3·43).

$$\left.\begin{array}{c} -V\sqrt{n}\,c_{n,n-1}+(2n-W)c_{n,n}-V\sqrt{n+1}\,c_{n+1,n}=0 \\ -V\sqrt{n+1}\,c_{n,n}+(2n+1-W)c_{n+1,n}-V\sqrt{n+1}\,c_{n+1,n+1}=0\,. \\ n=0,\ 1,\ 2,\ldots \end{array}\right\} \qquad (3\cdot42)$$

This system of equations is now sufficiently simple to be solved numerically.

In Fig. 3 the eigen-solution c's are plotted for various values of the coupling constant V. From the figure one sees that by introducing a proper parameter V' the plots for c's can be approximated by the square root of the Poisson formula

$$c_m = e^{-(V'^2/2)}\,\frac{V'^m}{\sqrt{m!}} \qquad (3\cdot43)$$

where c_m has the meaning

$$c_m = \begin{cases} c_{m/2,\ m/2} & \text{for even } m \\ c_{(m+1)/2,\ (m-1)/2} & \text{for odd } m \end{cases} \qquad (3\cdot44)$$

The points connected by broken lines represent this approximation.

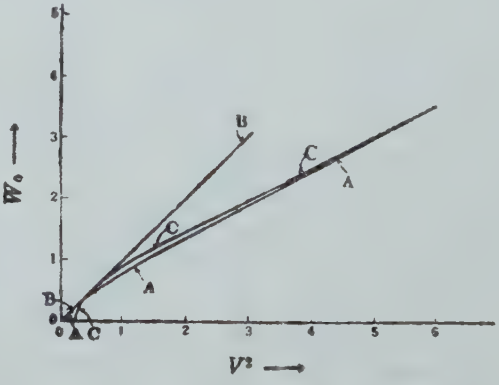


Fig. 4. Eigen-value W as function of V^2. AA: Exact curve. BB: According to (3·45). CC: According to (3·46).

Fig. 4 shows how the eigen-value W depend on V. For $V \ll 1$ it is approximately given by

$$W = -V^2, \qquad (3\cdot45)$$

and for $V \gg 1$ by

$$W = -\frac{1}{2}V^2 - \frac{1}{2} + \frac{m^2}{2V^2}. \qquad (3\cdot46)$$

From (3·45) we obtain E which, of course, agree with that given by (3·32). It is because in this case of weak coupling our substitution $\alpha=0$ was correct one. On the other hand, when the coupling is strong, the correct values of α and β are given by (3·37) and not zero, the result obtained by putting $\alpha=\beta=0$ is obviously less accurate. This E is obtained evidently by putting $\alpha=\beta=0$ in (3·36):

$$E=-\frac{1}{2}l^2\boldsymbol{K}_1^3-\frac{1}{2}\frac{\boldsymbol{K}_1^3}{\boldsymbol{K}_2^2}+\frac{\boldsymbol{K}_1^4}{2l^2\boldsymbol{K}_2^4}m^2. \tag{3·37}$$

The curve A in Fig. 4 represents W as a function of V obtained by solving (3·42) numerically. The figure shows how the curve A connects the straight line B representing (3·45) and the curve C representing (3·46). The figure indicates especially how the result of the perturbation theory, which is represented by the straight line B, becomes bad as the coupling constant V becomes large.

In Fig. 5 a physically interseting quantity, the "probability of dissociation," is given as a function of V. By this term is meant the probability that the proton/neutron is dissociating into the neutron/proton and mesotrons. This is given by

$$P=\frac{\sum c_{n+1,n}^2}{\sum c_{n,n}^2+\sum c_{n+1,n}^2} \tag{3·48}$$

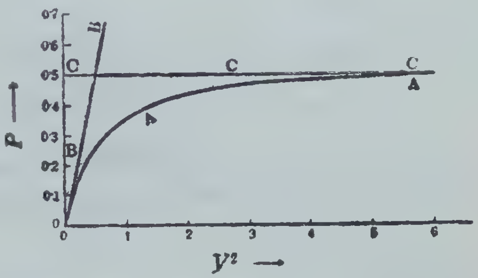


Fig. 5. Probability of dissociation as function of V^2. AA: Curve obtained numerically. BB: Values obtained by the perturbation theory. CC: Asymptotic value according to Wentzel.

The perturbation theory will give for this probability

$$P=V^2, \tag{3·49}$$

and according to Wentzel's method we have

$$P=\frac{1}{2}. \tag{3·50}$$

The figure shows how our result connect these two extreme cases and especially how rapidly the approximate value obtained by the perturbation theory becomes bad when the coupling becomes strong.

We shall here add few words about the meaning of α and β when the coupling is weak. For this purpose we solve the Schrödinger equation (3·4) in the limiting case of small l by means of the ordinary perturbation theory: in this case we may first neglect all higher φ and ψ in (3·4) retaining only φ_0, ψ_1 and φ_2. Then we have to solve the eigen value problem

$$\left.\begin{aligned}&K_0\varphi_0=-l\int\phi_1(\mathfrak{k}_1^+)\,g(k_1^+)\,d\mathfrak{k}_1^+\\&(K_0-K_1^+)\phi_1(\mathfrak{k}_1^+)=-l\left\{g(k_1^+)\varphi_0-\int\varphi_2(\mathfrak{k}_1^+\,;\,\mathfrak{k}_1^-)g(k_1^-)\,d\mathfrak{k}_1^-\right\}\\&(K_0-K_1^+-K_1^-)\varphi_2(\mathfrak{k}_1^+\,;\,\mathfrak{k}_1^-)=lg(k_1^-)\phi_1(\mathfrak{k}_1^+)\end{aligned}\right\}\quad(3\cdot51)$$

This gives by the ordinary perturbation calculation

$$\left.\begin{aligned}&K_0=-l^2\boldsymbol{K}_1^3+l^4\int d\mathfrak{k}\frac{g(k)^2}{K^2}\int g(k')^2\left\{\frac{1}{K+K'}-\frac{1}{K'}\right\}d\mathfrak{k}'\\&\varphi_0=1\\&\phi_1(\mathfrak{k}_1^+)=\frac{lg(k_1^+)}{K_1^+-l^2\int g(k)^2\left\{\frac{1}{K_1^++K}-\frac{1}{K}\right\}d\mathfrak{k}}\\&\varphi_2(\mathfrak{k}_1^+\,;\,\mathfrak{k}_1^-)=\frac{lg(k_1^+)}{K_1^+-l^2\int g(k)^2\left\{\frac{1}{K_1^++K}-\frac{1}{K}\right\}d\mathfrak{k}}\cdot\frac{lg(k_1^-)}{K_1^-+K_1^+-l^2\int\frac{g(k)^2}{K}\,d\mathfrak{k}}\end{aligned}\right\}$$

(3 · 52)

Comparing this results with our Hartree approximation :

$$\left.\begin{aligned}&\varphi_0=1\\&\phi_1(\mathfrak{k}_1^+)=\frac{c_{1,0}}{K_2^2}\,\frac{g(k_1^+)}{K_1^+-a}\\&\varphi_{\cdot}(\mathfrak{k}_1^+,\,\mathfrak{k}_1^-)=\frac{c_{1,1}}{K_2^3L_2^2}\,\frac{g(k_1^+)}{K_1^+-a}\cdot\frac{g(k_1^-)}{K_1^--\beta}\,,\end{aligned}\right\}\quad(3\cdot53)$$

we see that a and β have the meaning

$$\left.\begin{aligned}a&=l^2\left[\int g(k)^2\left\{\frac{1}{K_1^++K}-\frac{1}{K}\right\}d\mathfrak{k}\right]_{Av}\\\beta&=\left[K_1^+-l^2\int\frac{g(k)^2}{K}\,d\mathfrak{k}\right]_{Av}\end{aligned}\right\}\quad(3\cdot54)$$

where $[\quad]_{Av}$ means that K_1^+ in it is to be averaged in a proper way.

It is here to be noted that a tends to zero when l becomes vanishingly small, so a is very small when the coupling is weak. On the other hand, β tends to a quantity of the order of magnitude of $\boldsymbol{K}$ in this limit, so β is very large when the coupling is weak. These facts can be directly seen from (3 · 54).

E :—Metastability of proton-isobars.

Our result (3 · 38) indicates that, when the coupling is sufficiently strong, the states with $|m| \geqq 3/2$ will be stable. As mentioned before, this is because the interval between rotational levels

$$\varDelta E = \frac{1}{2l^2\boldsymbol{K}_3}\left\{m^2-(m-1)^2\right\} = \frac{1}{l^2\boldsymbol{K}_3}\left(m-\frac{1}{2}\right) \qquad (3 \cdot 55)$$

can become so small that it satisfies

$$\varDelta E < \kappa$$

when $l^2\boldsymbol{K}_3$ is sufficiently large.

Now, we notice that in our approximation the wave functions have the form

$$\left.\begin{aligned} f_+(\mathfrak{k}) &= \frac{g(k)}{K-\alpha}, \quad f_-(\mathfrak{k}) = \frac{g(k)}{K-\beta} \\ \alpha &= \frac{1}{l^2\boldsymbol{K}_3}\left(m-\frac{\boldsymbol{K}_1^3\boldsymbol{K}_3}{\boldsymbol{K}_2^4}\right), \quad \beta = \frac{1}{l^2\boldsymbol{K}_3}\left(-m-\frac{\boldsymbol{K}_1^3\boldsymbol{K}_3}{\boldsymbol{K}_2^4}\right). \end{aligned}\right\} \qquad (3 \cdot 56)$$

We see that when m is positive, β is always negative but α will be positive for $m \geqq 3/2$. As the consequence, there arises the possibility that the denominator of $f_+(\mathfrak{k})$ vanishes when α is larger than κ. That the denominator of the wave function vanishes at $K=\alpha$ means that in such a state the waves extend to infinity in space, so that the mesotrons are not bound to the nuclear particle but leak out with the energy α, or strictly speaking, with the average energy α, since we are here dealing with the Hartree approximation. Denoting this average energy of the mesotrons by $[K]_{Av}$, we have

$$[K]_{Av} = \alpha. \qquad (3 \cdot 57)$$

When m is negative the same situation occurs, β playing now the rôle of α.

When now the unstable state with the quantum number m had only one possibility of decaying into the state of $(m-1)$, emitting thereby only a single positive mesotron, this decay mesotron should have the energy $\varDelta E$ given by (3 · 55). But since we have $\boldsymbol{K}_1^3\boldsymbol{K}_3/\boldsymbol{K}_2^4 \simeq 4/3 > 1/2$, the average kinetic energy of (3 · 57) is not equal to this $\varDelta E$ but it is smaller than $\varDelta E$:

$$[K]_{Av} < \varDelta E. \qquad (3 \cdot 58)$$

What will now this inequality mean? This inequality implies, namely, that our unstable state with the quantum number m can not be interpreted as simply decaying into the state with the quantum number $(m-1)$ plus one free positive mesotron. Thus we must conclude that, when ΔE of $(3 \cdot 55)$ is larger than $\varkappa$, the state under consideration does not necessarily decay into the next lower state emitting one single mesotron, but this unstable state will decay into various end states emitting several mesotrons with various energies. The average of these energies is thereby given by $(3 \cdot 57)$.

Of couse, when α is too large compared with $\varkappa$, our Hartree approximation can give no reliable result. But when the coupling is such that, though ΔE is larger than $\varkappa$, α is still smaller than $\varkappa$, then our approximation may be still regarded as sufficiently good. In such a case, the fact that α is still smaller than $\varkappa$ means that the true wave function can still be approximated well by a wave function which does not extend to infinity. Thus such a state is meta-stable and have considerablly long life, although the excitation energy ΔE is larger than $\varkappa$.

Thus the probable existence of metastable proton-isobars is concluded in agreement with our conclusion obtained already in § 2D. There we have derived this result by means of the considerations based on the wave (or oscillator) picture and here by means of the considerations based on the particle picture of our system.

It is desirable to treat the scattering of a mesotron by some method also in this case of intermediate coupling. But unfortunately we have as yet found no adequate means to be applied to this problem.

(to be continued)

Department of Physics,
Tokyo Bunrika Daigaku.

Progress of Theoretical Physics Vol. II, No. 1, Jan.~Feb. 1947.

On the Theory of Ferromagnetism at 0°K.

By Syôhei MIYAHARA.

(Received Nov. 20, 1946)

(1) Introduction.

In the theory of ferromagnetism there exist two different standpoints, i. e. the phenomenalistic and the substantial. The former has, as Weiss' theory typically shows, the virtue of explaining quantitatively the relation between magnetism and temperature, or other quantities such as Curie-Weiss' constant and the anomalous specific heat etc. Heisenberg's theory must also be considered as a phenomenalistic one notwithstanding its electron-theoretical treatment, because it is unable to explain satisfactorily the appearance of ferromagnetism in connexion with the construction of substance in itself.

On the contrary, the virtue of the substantial method is related, as Slater's theory(1) typicaly shows, to what is the weakness of the other ; though it has the drawback of presenting difficulties in obtaining quantitative results, esp. the temperature dependencies. Notwithstanding this weak point we can expect various useful results from the theory, because the information concerning the electronic structure of solids is rapidly increasing nowadays, and ferromagnetic substances are specified by the appearance of ferromagnetism at the zero point of temperature, and the Curie-point has only a secondary meaning in spite of its practical importance.

Now we shall show several formal results concerning the ferromagnetism at 0°K, obtained under the assumption that the exchange-integrals and the level density (or distribution) function are entirely known.

(2) General theory of spontaneous magnetization at 0°K.

At 0°K the absolute entropy must be zero, so we do not need any information of the electronic complexion, but the lowest energy state only. Let U be the energy difference between the magnetic $(S\neq 0)$ and nonmagnetic ($S=0$) states at 0°K, where S means the total spin quantum number of the electronic system in consideration: $U<0$ results in the ferromagnetic case and $U>0$ the nonmagnetic. U is generally considered to be composed of two parts: one is the so-called binding energy U_1, that is simple summation of energies of total electrons, calculated directly from the level distribution function of one electron solution and the assumption as to the manner of electronic occupation to the levels. The other is the exchange energy, which is deduced from the electronic interaction and depends on S. Then U_1 is given by the following expression*:

---

(1) Slater. Phys. Rev. **49** (1936), 537.

$$U_1=\int_0^{E_1} EdZ(E)+\int_0^{E_2} EdZ(E)-2\int_0^{E_0} EdZ(E)\,, \tag{1}$$

where E is the level energy, E_1 and E_2 are respectively the highest energies of doubly and singly occupied levels in the magnetic state, and E_0 is the highest energy of (doubly occupied) levels in the nonmagnetic state, and $Z(E)$ means the level distribution function: the total number of levels having energies less than E is $Z(E)$. Then E , E_2 and E_0 are given by the following equations.

$$N=2Z(E_0)\,, \tag{2}$$

$$-S=\int_{E_1}^{E_2} dZ(E)\,. \tag{3}$$

The other part of the energy difference U_2 is not expressed in strict form because of the variety of exchange integrals, which depend on the wave number vectors k and k' of the related wave functions; then, taking the mean value, in accordance with Slater, $J_{kk'}$ is expressed by J/L, where J is the exchange integral in an isolated atom and L is the total number of atoms (or atomic cells) in the solid. Then

$$U_2=-JS^2/L\,, \tag{4}$$

and

$$U=U_1+U_2=\int_0^{E_1} EdZ(E)+\int_0^{E_2} EdZ(E)-2\int_0^{E_0} EdZ(E)-JS^2/L\,. \tag{5}$$

---

* **Integrals mean Stieltjes'.**

Let μ be the Bohr-magneton, the magnetic moment M of the solid is

$$M=-2S\mu. \tag{6}$$

For simplicity we use

$$n=N/L,\ y=M/L\mu,\ z(E)=Z(E)/L, \text{ and } u=U/L,$$

then

$$u=\int_0^{E_1} E\,dz(E)+\int_0^{E_2} E\,dz(E)-2\int_0^{E_0} E\,dz(E)-Jy^2/4 \tag{7}$$

$$y=\int_{E_1}^{E_2} dz(E), \tag{8}$$

$$n(=\text{const.})=z(E_1)+z(E_2)=2z(E_0), \tag{9}$$

and from (8) and (9),

$$y/2=\int_{E_1}^{E_0} dZ(E) \tag{10}$$

$$=\int_{E_0}^{E_2} dZ(E). \tag{11}$$

The above expressions (10) and (11) will also be expressed by

$$y=f(E_1\,;\ E_0)\quad [\text{or } y=f(E_2\,;\ E_0)]. \tag{12}$$

Now we must define the inverse function $E_1(\text{or } E_2)=f^{-1}(y\,;\ E_0)$. Generally $f(E_1)$ [or $f(E_2)$] has a finite number of discontinuous points $E_1=E',\ E'',\ldots,$ $[y'_-=f(E'-0),\ y'_+=f(E'+0)$ etc.], or $f(E_1)$ may be equal to a constant y^i for $E^i\leqq E_1<E^{i+1}$; so it is natural to define

$$\left.\begin{aligned} f^{-1}(y\,;\ E_0)&=E' \quad \text{for}\quad y'_-\leqq y<y'_+,\\ f^{-1}(y\,;\ E_0)&=E^i \quad \text{for}\quad y=y^i. \end{aligned}\right\} \tag{13}$$

$f^{-1}(y)$ is discontinuous at $y=y^i$ in the general case.

With the above definition u may be considered as a continuous function of y (notwithstanding the discontinuity of u as a function of E).

Now we shall consider the function $u(y)$. Taking the Pauli-principle into account, Ly or $-2S$ must be always less than the number of empty levels, then $u(y)$ is defined on the domain $0\leqq y\leqq y_{\max}$. (For instance by

Ni $y_{\max}=0.6$). Let y^* be the value of y which makes $u(y)$ smallest, then the magnetic moment of the system at 0°K is $Ly^*\mu$.

(3) The case of metals.

In the case of metals, we can suppose that $u(y)$ is differentiable in the domain and $u'(y)$ might also be continuous: therefore

$$y^*=0 \quad \text{or} \quad y_{\max}; \tag{14}$$

or $u(y^*)$ is minimum, therefore

$$u'(y)=0 \tag{15}$$

or

$$E_2-E_1-Jy=0, \quad \text{(asterisk is removed)}. \tag{16}$$

The minimum point y^* and the corresponding E_1 and E_2 will be found in the roots of the simultaneous transcendental equations (10), (11) and (16). There we can easily see that $y=0$ is always one of the roots of the equations; therefore, $u''(y)<0$ or from (7), (10) and (11),

$$J-1/\nu(E_0)>0 \tag{17}$$

is a sufficient condition for appearance of ferromagnetism, where $\nu(E)$ means the level density function $z'(E)$. In the case of Ni,[2] $J=7466\ \text{cm}^{-1}$ and $1/\nu(E_0)=5490\ \text{cm}^{-1}$, then the condition (17) is satisfied.

Next we shall consider the change of the magnetic moment at 0°K with respect to the change of the total number of electrons. Such a consideration must be useful to the theory of spontaneous magnetization of alloys. The value of y^* itself is difficult to express exiplicitly, but dy^*/dn is easily obtained.

i) When y^* coincides with the maximum value $y_{\max}$ of y,

$$dy^*/dn=-1 \tag{18}$$

ii) When y^* is a root of $u'(y)=0$, from (10), (11) and (16)

$$\frac{dy^*}{dn}=\frac{-[\nu(E_1)-\nu(E_2)]}{\nu(E_1)+\nu(E_2)-2J\nu(E_1)\nu(E_2)}. \tag{19}$$

---

(2) Slater, l. c.

If we assume $\nu(E_2)=0$ in the above expression, we have again (18). The result (18) is the same as that which Hirone's theory of Ni–alloys(3) gives, but the above method of deduction is more general.

(4) Application to body-centred and face-centred iron.

Recently we had an opportunity to know of the valuable works of Manning and Greene(4) concerning the electronic energy bands of body-centred and face-centred iron. From these works, we have

$$\text{for b.c. iron} \quad y_{\max}=2.25, \; 1/\nu(E_0)=13.7\cdot10^3\,\mathrm{cm}^{-1}$$

$$\text{for f. c. iron} \quad y_{\max}=2.22, \; 1/\nu(E_0)=16.7\cdot10^3\,\mathrm{cm}^{-1}$$

As the f. c. iron might be nonmagnetic at 0°K, J must satisfy the inequalities

$$13.7\cdot10^3\,\mathrm{cm}^{-1}<J<16.7\cdot10^3\,\mathrm{cm}^{-1}. \tag{20}$$

If we assume the lowest value for J, we obtain $y^*=2.2$, from (10), (11), (16) and the curve of the $\nu(E)$, and this value is very satisfactory.

The validity of the relation (19) is not so good for Fe as in the case for Ni.

Physical Institute,
Nagoya Imperial University.

---

(3) Hirone, Sc. Rep. Tohoku Univ. **27** (1938), 101.

(2) Manning, Phys. Rev. **63** (1943), 190; Greene, Mannig, Phys. Rev. **63** (1943), 203.

Letters to the Editor

The Self-Energy of the Electron and the Mass Difference of Nucleons

It has been made clear(1) that Bopp's new electromagnetic theory(2) turns out to be, on quantization, a mixed theory of Maxwells electromagnetic field and the neutral vector Yukawa field. As a sequel, however, the following two new difficulties present themselves. Firstly, the energy of the meson becomes negative, and secondly, the self-energy diverges due to the fluctuation of the field. Inoue and Takagi advocate the adoption of a scalar field as the Yukawa field, in order to remove the first difficulty(3). We desire to point out that the second difficulty, too, can be obviated by this combination of fields.

The self-energy of an electron at rest due to an electromagnetic field becomes, on calculating positron-theoretically in accordance with Weisskopf(4),

$$W_E=\frac{e^2}{\hbar c}\,\frac{mc^2}{\pi}\left(\frac{3}{2}\lim_{K\to\infty}\log K+\frac{3}{2}\log 2-\frac{1}{2}\right) \quad (1)$$

where m denotes the mass of the electron, e its charge, $2\pi\hbar$ Planck's constant, c the light velocity. This is the sum of the electrostatic self-energy and the electrodynamic one due to the fluctuation of the field, and differs from the result of the ordinary one-body problem calculation in that it diverges logarithmically. Carrying out exactly similar calculations for the self-energy due to the meson field, we obtain

$$W_M=-\frac{f^2}{\hbar c}\,\frac{mc^2}{2\pi}\left(\frac{3}{2}\lim_{K\to\infty}\log K-\frac{3}{2}\log\frac{m_u}{m}+\frac{1}{4}-\frac{1}{2}\log 2\right) \quad (2)$$

Here we have considered, just as Inoue and Takagi, only the scaler type of interaction between the electron and the meson field, and denoted its constant by f. Further, m_u represents the mass of the meson.

The expression (2) diverges, just as (1), logarithmically, and if we postulate the relation

$$e^2=f^2/2$$

between the constants of the interactions, the result becomes finite, giving as the total self-energy of the electron

$$W_e=W_E+W_M=\frac{e^2}{\hbar c}\,\frac{mc^2}{\pi}\left(\frac{3}{2}\log\frac{2m_u}{m}+\frac{1}{2}\log 2-\frac{3}{4}\right)$$

If we take $m_u \sim 200$ m, this value is so small compared with the electron mass, that the additional mass due to the self-energy may, in general, be neglected.

As to the theories obviating the divergence of self-energy, there is, besides those following the line of Born and Bopp, which we have examined, another originating in Wentzel(5) and developed by Dirac and Pauli.(6) This theory employs manipulation called the λ–limiting process, together with a new method of quantization involving the introduction of negative energy photons, and is claimed by Pauli to be capable of removing all the divergences in the quantum theory of fields. In particular, as concerns self-energy, the λ–limiting process is instrumental in eliminating the electrostatic part, while the new method of quantization is effective in anulling that part due to the fluctuation of the field. But this is the result of a one body-problem type of computation, and if the correct, positron-theoretical method of calculation is followed, the self-energy diverges once more and neither the λ–limiting process nor the new quantization is of any avail here. As Pauli says, although this theory is extremely unsatisfactory in that it simultaneously employs two altogether independent methods, and further requires a reinterpretation of the results, there is something in it that reminds us of the eve of the advent of the quantum mechanics. But, it must be admitted to be a fatal defect of their theory that it loses sight of the solution of the self energy problem, which was its original subject matter.

Our theory can be directly extended to protons as well, and gives, as its self-energy, the expression

$$W_p = -\frac{e^2}{\hbar c}\,\frac{Mc^2}{\pi}\left(\frac{3}{4}-\frac{m_u}{M}\tan^{-1}\frac{M}{m_u}\right)$$

where M denotes the intrinsic mass of the proton, and it is assumed that $M > m_u > m$. $\Delta M = W_p/c^2$ may be regarded as giving the mass difference between neutrons and protons, and, if we take $m_u \sim 200$ m, we obtain $\Delta M \sim -2\,m$, which is approximately of the same order as the observed value. Our theory may be claimed noteworthy as there seems to be, at present, no other capable of explaining this ΔM.

S. Sakata
O. Hara

Institute of Theoretical Physics,
Nagoya Imperial University.
April 8, 1947.

References

(1) S. Sakata. Progr. Theor. Phys. 2 (1947), in press.
O. Hara. Lecture at the annual meeting of the Japanese Physical Society (1946).
(2) Bopp. Ann. d. Phys. 38 (1940), 345.
(3)* T. Inoue & S. Takagi. Kagaku, 16 (1946), 205.
In this theory, it is assumed $e^2=f^2$.
(4) Weisskopf. Phys. Rev. 56 (1939), 72.
(5) Wentzel. ZS. Phys. 86 (1933), 479.
(6) Pauli. Rev. Mod. Phys., 15 (1943), 175.

On the Lifetime of the Neutral Meson

The present author has shown that the radiation damping gives the finite transition probabilities for such processes that the virtual pairs of electrons or nucleons are produced in the intermediate states and whose transition probabilities are usually divergent according to the ordinary perturbation theory.

For example, we calculated the two photons disintegration of a neutral pseudoscalar meson and the three photons disintegration of a neutral vector meson.(1)

We obtained the decay formula

$$\frac{1}{\tau_0}=\frac{1}{36}\left(\frac{e^2}{\hbar c}\right)^{-2}\frac{f_2^2}{\hbar c}\,\frac{\mu c^2}{\hbar}\qquad\text{(pseudoscalar)}$$

$$\frac{1}{\tau_0}=\frac{1}{216}\left(\frac{e^2}{\hbar c}\right)^{-1}\frac{g_1^2}{\hbar c}\,\frac{\mu c^2}{\hbar}\qquad\text{(vector)},$$

where μ is a meson mass, f_1 and g_1 are the interaction constants of the meson with the nucleon. It is noticable that these results are free from any divergent integral.

Usiug the usual values of constants, we get

$$\tau_0=4.2\times10^{-24}\ \text{sec.}\qquad\text{(pseudoscalar)}$$

$$\tau_0=3.4\times10^{-21}\ \text{sec.}\qquad\text{(vector)}$$

Detailed account will be given in the later issur of this journal.

Y. Tanikawa

Institute of Theoretical Physics,
Nagoya Imperial University.
April 15, 1947.

(1) S. Sakata and Y. Tanikawa: Phys. Rev., 57 (1940), 548.
Y. Tanikawa: Proc. Phys.-Math. Soc. Japan. 24 (1942), 610.

On the Decay of the Charged Meson

Sakata has suggested a modified theory of the meson decay(1) which would remove the well known discrepancy between the theoretical and the experimental lifetime of a charged meson. His results however depend on the divergent integrals which make any result unreasonable. Then, we calculated the same decay-process by taking account of the radiation damping which gives the divergent free results.

We obtained the decay formula of the charged meson

$$\frac{1}{\tau_0}=\frac{4}{\mu}\left(\frac{\mu}{m}\right)^{-5}\left(\frac{\hbar}{\mu c^2}\right)^{-2}\left(\frac{1}{4}\,\frac{g_1{}^2}{\hbar c}\,T_1+\frac{g_2{}^2}{\hbar c}\,T_2\right) \quad (1)$$

for a vector meson, and

$$\frac{1}{\tau_0}=\frac{2}{\pi}\left(\frac{\mu}{m}\right)^{-5}\left(\frac{\hbar}{\mu c^2}\right)^{-2}\frac{f_1{}^2}{\hbar c}\,T_1{}' \quad (2)$$

fos a pseudoscalar meson, where m is an electron mass and μ a meson mass, g_1, g_2, and f_1 denote the interaction constants of meson with nucleon, T_1, T_2 and $T_1{}'$ the time constants of the beta-decay.

Adopting the usual values of constants, we have

$$\tau_0\ (\text{vector}) \simeq 10^{-37}\ \text{sec.} \quad (3)$$

Then, we see that the descrepancy of the lifetime for a vector meson increases in the modified theory proposed by Sakata rather than in the original Yukawa theory. The same is true for a scalar and a pseudoscalar meson as easilly seen.

On the other hand, Yukawa's theory too is not free from this sort of difficulty, because there exists always Fermi's type of the direct interaction between the heavy and the light particles⁽²⁾ by which the modified prosess of the meson decay is brought forth.

It still seems that there exists only one possibility in the pseudoscalar meson theory which get rid of the difficulty, since in this case we can determine the interaction constants for the meson decay and the beta-decay separately. In fact, if we take in (2)

$$\frac{f_1{}^2}{\hbar c}\cdot T_1{}' \simeq 10^{-29}\ \text{sec.} \quad (4)$$

we obtain

$$\tau_0\ (\text{pseudoscalar}) \simeq 10^{-6}\ \text{sec.}$$

It therefore seems to be concluded that the charged meson which can be observed in cosmic rays is only of the pseudoscalar type.

Detailed account will be given in the later issue of this journal.

Y. Tanikawa
Institute of Theoretical Physics,
Nagoga Imperial University.
April 15, 1947.

(1) S. Sakata: Proc. Phys.-Math. Soc. Japan, 23 (1941), 283.

(2) N. Kemmer: Proc. Roy. Soc.. **166** (1938), 127.

Progress of Theoretical Physics Vol. II, No. 2, Mar ~Jun., 1947.

Elastic Collision Cross Section of Oxygen Atom for Slow Electron.

By Takahiko YAMANOUCHI.

(Received Nov. 22, 1946)

1. Introduction.

According to quantum mechanics, the effective cross section of an atom for the elastic collision of electron is given by⁽¹⁾

$$Q=\frac{4\pi}{k^2}\sum_{l=0}^{8}(2l+1)\sin^2\delta_l\,, \tag{1}$$

where k is the velocity of the electron in atomic units, and δ_l is the phase-shift of the l-th spherical wave into which the wave function of the colliding electron is decomposed. Integrating the wave equation for the radial part $y(r)$ in an appropreate atomic field $U(r)$, and joining it smoothly to $c_1F_l(kr)+c_2G_l(kr)\,(F_l(z)=\sqrt{\pi z/2}\,J_{l+(1/2)}(z),\ \ G_l(z)=\sqrt{\pi z/2}\,Y_{l+(1/2)}(z))$ at $r=r_0$ where $U(r)$ vanishes practically, the phase-shift δ_l is determined from the asymptotic form:

$$y(r)\sim C\sin(kr-l\pi/2+\delta_l)$$

by the relation:

$$c_1=C\cos\delta_l,\quad c_2=C\sin\delta_l\,. \tag{2}$$

At the same time, the normalizing factor that makes the amplitude of the wave function unity at the infinity is obtained from

$$N_l=1/C=1/\sqrt{c_1^2+c_2^2}\,. \tag{3}$$

For $k\to0$, in particular, we have

$$\delta_l=\alpha k^{2l+1},\quad N_l=\beta k^{l+1}, \tag{4}$$

---

(1) Mott-Massey, "*Theory of Atomic Collision*", II, § 1, 24.

as can readily be shown by the series expansion of Bessel functions. δ_0 only will therefore contribute to Q for small value of k.

2. Cross Section for Oxygen Atom.

For the mean atomic field $U(r)$, the self-consistent field with exchange$^{(2)}$ of the ground state $1s^2\ 2s^2\ 2p^4\ {}^3P$ of oxygen atom is adopted, for which $U(r)$ vanishes in significant figures at $r=8$. For $l=0$ and $l=2$, the wave functions and phase-shifts have already been calculated for another purpose$^{(3)}$. The newly calculated results for $l=1$ are shown in Fig. 1 and Fig. 2.

The coefficients α and β in (4) are as follows:

l	0	1	2
α	2.62	1.617	0.295
β	14.15	15.63	0.358

Fig. 3, 4, and 5 show the variation of $\tan\delta_l/k^{2l+1}$ and N_l/k^{l+1}, in which however δ_2/k^5 is omitted as it is practically 0. For the calculation of Q,

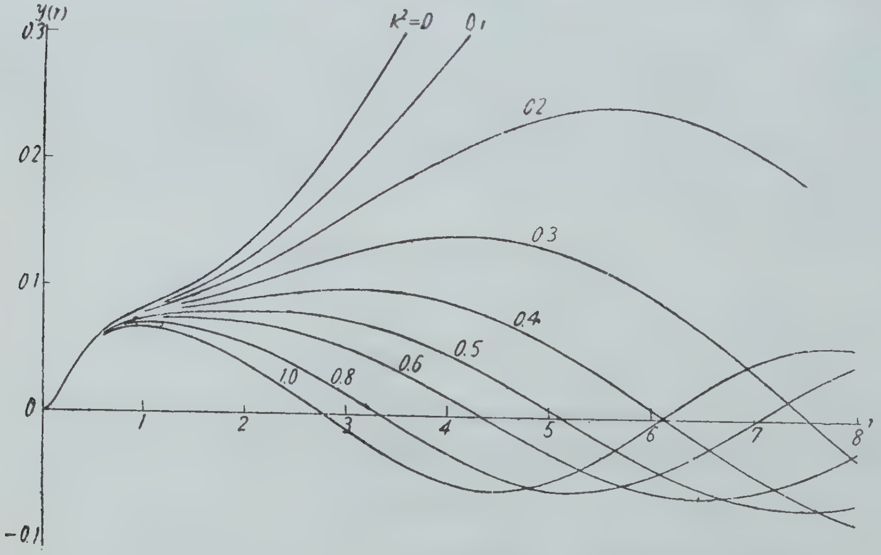


Fig. 1.

---

(2) Hartree, Hartree and Swirles, Phil. Trans. **A**, **238** (1939), 229.

(3) Yamanouchi, Proc. Phys.-Math. Soc. Japan, **22** (1940), 569.

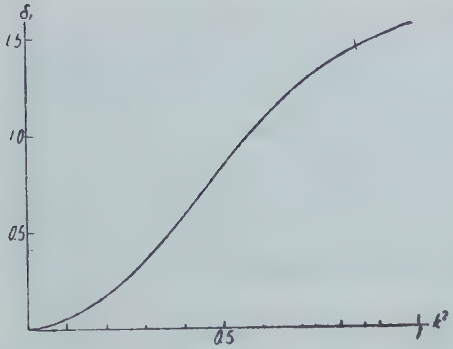


Fig. 2.

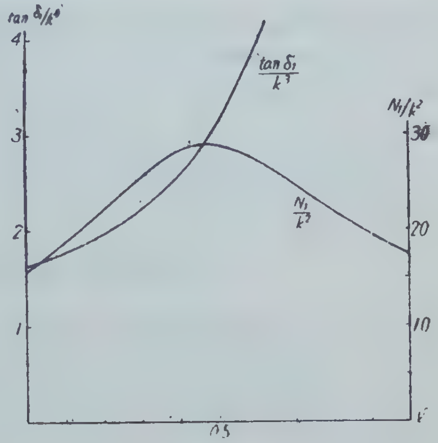


Fig. 4.

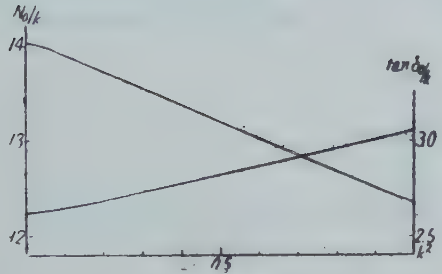


Fig. 3.

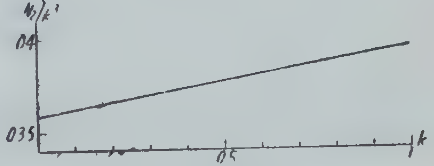


Fig. 5.

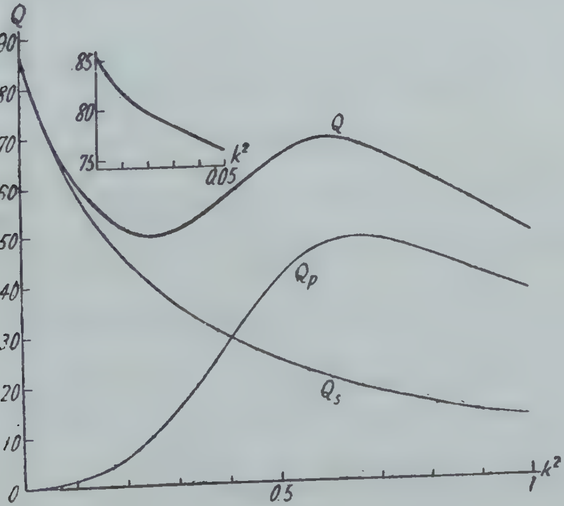


Fig. 6.

therefore, the first two terms will suffice.

The cross section (in a. u.) is shown in Fig. 6. The thin lines indicate the partial cross section Q_s and Q_p due to $l=0$ and $l=1$, and the thick line the total cross section. The reason that the curve for $l=1$ has a maximum at $k^2=0.65$ has been explained in our former paper[(4)].

For $k \to 0$, we have approximately $\sin \delta_0 = 2.62\,(1-1.71\,k^2)$.

3. Collision Frequency in Ionosphere.

As an application of the above result, we shall consider the collision of oxygen atoms and free electrons in the ionosphere. Let the number of oxygen atoms in unit volume and the energy distribution function of the free electron be N and $f(\varepsilon)$ respectively. Then the collision frequency is given by

$$\nu = N \int_0^\infty kQ(\varepsilon) f(\varepsilon)\, d\varepsilon. \tag{5}$$

If we assume for $f(\varepsilon)$ the Maxwell distribution corresponding to the absolute temperature T,

$$\nu = \frac{2\sqrt{2}}{\sqrt{\pi}} \frac{N}{(\varkappa T)^{3/2}} \int_0^\infty \varepsilon Q(\varepsilon)\, e^{-\varepsilon/\varkappa T}\, d\varepsilon. \quad (\varkappa : \text{Boltzmann constant}) \tag{6}$$

In the ionosphere, where the mean kinetic energy of the electrons is of the order 0.003 in a. u., we may put $Q(\varepsilon)=Q(0)=Q$, and we have

$$\nu = 2NQ\sqrt{2\varkappa T/\pi}. \tag{7}$$

Substituting $Q=86.1$ (a. u.)$=24.1 \times 10^{-15}\,\text{cm}^2$, it gives

$$\nu = 0.245\, N\sqrt{T}\,(\text{a. u.}) = 1.5\, N\sqrt{T} \times 10^{-9} \quad (\text{c. g. s. units}),$$

and its numerical value is about 2×10^2 by assuming $N=4 \times 10^9$ (F layer), and $T=10^3$ K°.

The cross section for the nitrogen molecule has been calculated by Fisk[(5)], which for $k \to 0$ is equal to 24 in a. u. or $6.7 \times 10^{-16}\,\text{cm}^2$. If we assume the density of nitrogen molecules to be $2N$, we have for the mean collision frequency

---

(4) Yamanouchi, Inui and Amemiya, ibid. **22** (1940), 398.

(5) Fisk, Phys. Rev. **49** (1936), 167.

$$\nu = 2.26\, N\sqrt{T} \times 10^{-9}$$

which is about 3×10^2 for the above values of N and T. This agrees with the experimental value at least in its order of magnitudes, but it is perhaps too small. The collision with the other kinds of atoms and ions must be included.

Department of Applied Mathematics,
Second Faculty of Engineering,
Tokyo Imperial University.

Progress of Theoretical Physics Vol. II, No. 2, Mar.～Jun., 1947.

On the Interaction of Mesons with the Gravitational Field. I.

By Ryōyū UTIYAMA.

(Received Feb. 8, 1947)

Introduction.

The theory of the interaction of electrons with the gravitational field was discussed by Weyl,(1) Fock,(2) Schrödinger(3) and Rosenfeld,(4) and that of the electromagnetic field with the latter was discussed by Rosenfeld(5) and Solomon.(6) In these papers almost anything was not considered about the freedom of the functions of the gravitational field. And these authors only used that the gravitational wave is a transversal one having two components as shown by Einstein.(7)

While, the relationship of the freedom of the field-functions with the spin of the quanta acompanied by the field was discussed in general 1939,(8) and it was pointed out especially by Pauli(9) and Fierz(10) that the equations of the field with the rest mass zero and the spin 2 are equal to those of the gravitational field in the 1st approximation given by Einstein.(11) So we tried to treat the interaction of the quanta (gravitational quanta or gravitons)

---

* This paper was published at the meeting of Physico-mathematical Society of Japan in April 1941 and October 1942. But because of the war the printing was delaied.

(1) H. Weyl: Z. S. f. Phys. **56** 330 (1929).

(2) V. Fock: Z. S. f. Phys. **57** 261 (1929).

(3) E. Schrödinger: Berl. Ber. Math-phys. Kl. (1932) 105.

(4) L. Rosenfeld: Ann. d. Phys, **5** 113 (1930).

(5) L. Rosenfeld: Z. S. f. Phys. **65** (1930) 589.

(6) J. Solomon: Journ. de. Phys. **9** (1938) 7.

(7) A. Einstein: Berl. Ber. (1916) 688 & (1918) 154.

(8) P. A. M. Dirac: Proc. Roy. Soc. (A) **155** (1936) 447.
M. Fierz: Helv. Phys. Acta. **12** (1939) 3.
M. Fierz & W. Pauli: Proc. Roy. Soc. (A) **173** (1939) 21.

(9) W. Pauli: Huitieme Conseil de Phys. Solvay (1939).

(10) M. Fierz & W. Pauli: cf. (8).

(11) A. Einstein: cf. (7).

of the rest mass zero and the spin 2 with the other elementary particles by considering that the above Einstein's field equations of the first approximation are exact. And here, for the simplicity we consider the neutral vector meson field interacting with the gravitational quanta. For if we put the meson mass zero, we get the equations of photons interacting with the latter which were formerly discussed by Rosenfeld.(12) Further, the interaction of the spinor field or the tensor field of higher rank with gravitons can be treated generally following these method used in this paper.

By the way, the method of obtaining the symmetric energy momentum tensor of any tensor or spinor field is given, and the equivalence of those with the canonical one is proved in general. The same problem was discussed in detail and more generally by K. Husimi,(13) who had shown in the region of the theory of the special relativity.

This paper is divided for convenience into (I) and (II), in (I) it was treated classically, and in (II) quantum-mechanically.

In conclusion, the author wishes to express his heartful thanks to Prof. Dr. K. Husimi for his valuable advices and kind interest for this work.

§ 1. General Theory.

i) Theory of Invariant Variation.(14)

Consider the field functions $Q_\alpha(x)$, $(\alpha=1, 2, \ldots N)$ and $Q_{\alpha,\mu}=\partial Q_\alpha/\partial x^\mu$, where $x^\mu=(x^1, x^2, x^3, x^4)=(x, y, z, ict)$ and Q_α is any spinor or tensor of the arbitrary rank. When the Lagrangian of a system is given by

$$\boldsymbol{L}(x, Q_\alpha, Q_{\alpha,\mu}),$$

consider the following integral

$$I=\int_V \boldsymbol{L}\,dx^1dx^2dx^3dx^4=\int_V \boldsymbol{L}\,dx\,, \tag{1}$$

where the integration is performed over any four dimensional volume V closed by three dimensional surface S. Now for r arbitrary functions

---

(12) L. Rosenfeld: cf. (5).

(13) K. Husimi: National Research Council of Japan **4** 81.

(14) E. Noether: Gött. Nachr. (1918) 235.
W. Pauli: Relativitätstheorie 617.

$\xi^\rho(x)(\rho=1,2\ldots r)$, we consider the following infinitesimal transformation

$$\left.\begin{aligned} \delta x^\mu &= x'^\mu - x^\mu = a^\mu_\rho\, \xi^\rho{}^{*} \\ \delta Q_\alpha - Q'_\alpha(x') - Q_\alpha(x) &= B_{\alpha,\rho}\xi^\rho + C^\mu_{\alpha,\rho}\frac{\partial \xi^\rho}{\partial x^\mu}, \end{aligned}\right\} \qquad (2)$$

where x'_μ and $Q'_\alpha(x')$ are the transformed coordinates and field functions at the world point x'_μ respectively, and a^μ_ρ, $B_{\alpha,\rho}$, $C^\mu_{\alpha,\rho}$ are functions of x^μ and Q. For the transformation (2), I is transformed to

$$I' = \int_{V'} \boldsymbol{L}'(x'\, Q'_\alpha(x')\, Q'_{\alpha,\mu}(x'))\, dx',$$

which is written as follows using the infinitesimal characters of (2);

$$I' = \int_V \boldsymbol{L}'\{x+\delta x,\ Q+\delta Q\ldots\}\, \frac{\partial(x^{1'},\, x^{2'},\, x^{3'},\, x^{4'})}{\partial(x^1 x^2 x^3 x^4)}\, dx.$$

If I is invariant for (2), we get

$$\delta I = I' - I = \int_V \left[\boldsymbol{L}'(x+\delta x,\ Q+\delta Q\ldots)\,\frac{\partial(x')}{\partial(x)} - \boldsymbol{L}(x,\ Q\ldots)\right] dx = 0.$$

If we put

$$\delta \boldsymbol{L} = \boldsymbol{L}'(x'Q') - \boldsymbol{L}(x,\, Q),$$

then

$$\delta I = \int_V \left\{\delta \boldsymbol{L} + \boldsymbol{L}\frac{\partial(\delta x^\mu)}{\partial x^\mu}\right\} dx = 0.$$

Defining the substantial variation of any field quantity Φ as follows;

$$\delta^* \Phi = \Phi'(x) - \Phi(x), \qquad (3)$$

then

$$\delta \Phi = \delta^* \Phi + \frac{\partial \Phi}{\partial x^\mu}\,\delta x^\mu,$$

using this relation, δI becomes;

---

* In this paper the summation notations are omitted following the tensor calculous.

$$\delta I=\int_V\left[\delta^*\boldsymbol{L}+\frac{\partial}{\partial x^\mu}(\boldsymbol{L}\delta x^\mu)\right]dx=0\,, \tag{4}$$

where

$$\delta^*\boldsymbol{L}=\frac{\partial\boldsymbol{L}}{\partial Q_\alpha}\delta^*Q_\alpha+\frac{\partial\boldsymbol{L}}{\partial Q_{\alpha,\mu}}\delta^*Q_{\alpha,\mu}\,. \tag{5}$$

If (4) is satisfied for the arbitrary ξ^ρ, and V it is necessary that

$$\delta^*\boldsymbol{L}+\frac{\partial}{\partial x^\mu}(\boldsymbol{L}\delta x^\mu)=0\,. \tag{6}$$

Inserting in (6) with (2), and using the notation

$$\frac{\delta\boldsymbol{L}}{\delta Q_\alpha}=\frac{\partial\boldsymbol{L}}{\partial Q_\alpha}-\frac{\partial}{\partial x^\rho}\frac{\partial\boldsymbol{L}}{\partial Q_{\alpha,\rho}}\,, \tag{7}$$

(6) becomes

$$\begin{aligned}&\frac{\delta\boldsymbol{L}}{\delta Q_\alpha}\delta^*Q_\alpha+\frac{\partial}{\partial x^\mu}\left\{\frac{\partial\boldsymbol{L}}{\partial Q_{\alpha,\mu}}\delta^*Q_\alpha+\boldsymbol{L}\delta x^\mu\right\}\\&=\left[\frac{\delta\boldsymbol{L}}{\delta Q_\alpha}(B_{\alpha,\mu}-Q_{\alpha,\rho}\,a^\rho_\mu)-\frac{\partial}{\partial x^\nu}\left(\frac{\delta\boldsymbol{L}}{\delta Q_\alpha}C^\nu_{\alpha,\mu}\right)\right]\xi^\mu\\&\quad+\frac{\partial}{\partial x^\nu}\left[\left\{\boldsymbol{L}a^\nu_\mu+\frac{\delta\boldsymbol{L}}{\delta Q_\alpha}C^\nu_{\alpha,\mu}+\frac{\partial\boldsymbol{L}}{\partial Q_{\alpha,\nu}}(B_{\alpha,\mu}-Q_{\alpha,\rho}\,a^\rho_\mu)\right\}\xi^\mu\right.\\&\quad\left.+\frac{\partial\boldsymbol{L}}{\partial Q_{\alpha,\nu}}C^\mu_{\alpha\lambda}\frac{\partial\xi^\lambda}{\partial x^\mu}\right]=0\,.\end{aligned} \tag{8}$$

Integrating (8) over V and assuming that ξ^ρ and its derivatives vanish on the boundary surface S, the integration of the divergence-term of (8) becomes zero, and we get the following identity as the necessary condition;

$$\frac{\delta\boldsymbol{L}}{\delta Q_\alpha}(B_{\alpha,\mu}-Q_{\alpha,\rho}\,a^\rho_\mu)-\frac{\partial}{\partial x^\nu}\left\{\frac{\delta\boldsymbol{L}}{\delta Q_\alpha}C^\nu_{\alpha,\mu}\right\}\equiv0\,. \tag{9}$$

If we introduce (9) into the 1st term of (8), the divergence-term must be zero. Considering that ξ^ρ and its derivatives of any order at any world point are independent, we get the several following identities;

$$\boldsymbol{E}^{\nu\mu}_\lambda=-\boldsymbol{E}^{\mu\nu}_\lambda \tag{10}$$

$$\boldsymbol{D}^{\nu}_{\lambda}+\frac{\partial}{\partial x^{\mu}}\boldsymbol{E}^{\mu\nu}_{\lambda}=0 \tag{11}$$

$$\frac{\partial \boldsymbol{D}^{\nu}_{\lambda}}{\partial x^{\nu}}=0\,, \tag{12}$$

where

$$\boldsymbol{D}^{\nu}_{\lambda}=\boldsymbol{L}a^{\nu}_{\lambda}+\frac{\delta \boldsymbol{L}}{\delta Q_{\alpha}}C^{\nu}_{\alpha\lambda}+\frac{\partial \boldsymbol{L}}{\partial Q_{\alpha,\nu}}(B_{\alpha,\lambda}-Q_{\alpha,\rho}\,a^{\rho}_{\lambda}) \tag{13}$$

and

$$\boldsymbol{E}^{\nu\mu}_{\lambda}=\frac{\partial \boldsymbol{L}}{\partial Q_{\alpha,\nu}}C^{\mu}_{\alpha\lambda}\,. \tag{14}$$

(12) is obtained from (10) and (11). (9) represents that N—equations of the field $(\delta \boldsymbol{L}/\delta Q_{\alpha})=0$ are not independent each other but some r equations can be derived from other $N-r$ equations. Therefore $N-Q's$ are not completely determined by field equations but contain r undetermined functions, and these can be eliminated by the transformation (2).

At the end of this paragraph we shall give some important formulae which are useful in (II). To give these formulae, we insert the field equations

$$\frac{\delta \boldsymbol{L}}{\delta Q_{\alpha}}=0\,,$$

into (8), and integrate it at any instant, $x^4=x^4_0$ over the three dimensional space $x^1x^2x^3$ using ξ^{ρ} to be zero over the boundary two dimensional surface, (by the partial integration), then we obtain

$$\int\Big[\boldsymbol{E}^{44}_{\lambda}\frac{\partial^2\xi^{\lambda}}{(\partial x^4)^2}+\Big\{\frac{\partial \boldsymbol{E}^{44}_{\lambda}}{\partial x^4}+\boldsymbol{D}^4_{\lambda}-\sum_{k=1}^{3}\frac{\partial}{\partial x^k}\boldsymbol{E}^{4k}_{\lambda}\Big\}\frac{\partial\xi^{\lambda}}{\partial x^4}$$
$$+\frac{\partial}{\partial x^4}\Big(\boldsymbol{D}^4_{\lambda}-\sum_{k=1}^{3}\frac{\partial}{\partial x^k}\boldsymbol{E}^{4k}_{\lambda}\Big)\xi^{\lambda}\Big]dx^1dx^2dx^3=0\,.$$

Putting each coefficient of each derivative of ξ^{ρ} equal to zero, then at any instant $x^4=x^4_0$

$$\boldsymbol{E}^{44}_{\lambda}=0 \tag{15.1}$$

$$\frac{\partial \boldsymbol{E}^{44}_{\lambda}}{\partial x^4}+\boldsymbol{D}^4_{\lambda}-\sum_{k=1}^{3}\frac{\partial \boldsymbol{E}^{4k}_{\lambda}}{\partial x^k}=0 \tag{15.2}$$

$$\frac{\partial}{\partial x^4}\left(D_\lambda^4-\sum_{k=1}^{3}\frac{\partial E_\lambda^{4k}}{\partial x^k}\right)=0\,. \tag{15·3}$$

Now we put as the initial condition

$$\frac{\partial E_\lambda^{44}}{\partial x^4}=0 \quad \text{at} \quad x^4=x_0^4\,, \tag{16}$$

then

$$D_\lambda^4-\sum_{k=1}^{3}\frac{\partial}{\partial x^k}E_\lambda^{4k}=0 \quad \text{at} \quad x^4=x_0^4\,. \tag{16}'$$

While we have the equation (15·3), so the relation (16) or (16′) are satisfied all the time and from (16) and (15·1), the relation (15·1) are also satisfied continuously at every time.

ii) Meson Field and Gravitational Field.

Consider the weak gravitational field described by the fundamental tensor $g_{u\nu}$, the deviation of which from $\delta_{\mu\nu}$ is very small compared to unity, and so

$$g_{\mu\nu}dx^\mu dx^\nu \approx \sum_{k=1}^{4}(dx^k)^2$$

Further we assume that

$$\det(g_{\mu\nu})=g$$

is positive definite. Now besides $g_{\mu\tau}$, it is convenient to consider the so-called "four legs" (*vier Beine*) $h_{i,\mu}$, where the index i represents the i–th leg, whereas the index μ represents the μ–th component of the vector with respect to the x^ν co-ordinate system.*

They are related to $g_{\mu\nu}$ as follows;

$$\left.\begin{array}{c} h_{i\mu}h^{k\mu}=\delta_i^k \quad h_{i\mu}h_\nu^i=g_{\mu\nu} \\ h_i^\mu h^{i\nu}=g^{\mu\nu} \end{array}\right\}, \tag{17}$$

the 1st relations represent the orthogonality, and at the same time give the definition of $h^{k\mu}$. The local co-ordinate system $h_{i\mu}$ at any world point (x^μ)

---

* Hereafter the Greek letters represent the components with respect to the x-coordinate system, on the other hand the Latin letters represent those referring to the four legs.

can be regarded as a Lorentz system in the neighbourhood of that world point, so the distinction of upper and lower positions of Latin indices are unnecessary (because of the use of the imaginary time coordinate). The relation of Latin and Greek indices of any vectors or tensors are given

$$\left.\begin{aligned} A_k &= h^{\mu}_{k} A_{\mu} \\ A_{\mu} &= h_{k\mu} A_k = h^{k}_{\mu} A_k \,. \end{aligned}\right\} \tag{18}$$

Now consider a meson field described by a real four vector U_μ and a real scalar V, the latter is introduced to remove some formal difficulties. The Lagrangian of the total system is given by

$$\boldsymbol{L} = \boldsymbol{M} - \frac{1}{8\pi\varepsilon^2}\boldsymbol{G}\,, \tag{19}$$

where

$$\boldsymbol{M} = h\left\{\frac{\varkappa^2}{2}(U_\mu - V_{,\nu})(U_\nu - V_{,\nu})g_{\mu\nu} + \frac{1}{4}U_{\mu\nu}U_{\alpha\beta}g^{\mu\alpha}g^{\nu\beta}\right\}, \tag{20}$$

and

$$\left.\begin{aligned} \boldsymbol{G} &= \boldsymbol{R} - 2\frac{\partial}{\partial x^\nu}\left\{h^{\nu}_{e}\frac{\partial(h^{\sigma}_{e}h)}{\partial x^{\sigma}}\right\} \\ \boldsymbol{R} &= h\Big\{\eta_{e[\rho\sigma]}\eta_{k[\mu\sigma]}h^{\rho}_{e}h^{\mu}_{k}g^{\sigma\nu} \\ &\qquad -\frac{1}{2}\eta_{e[\rho\sigma]}\eta_{k[\mu\nu]}h^{\rho}_{e}h^{\mu}_{k}g^{\sigma\nu} \\ &\qquad -\frac{1}{4}\eta_{e[\rho\sigma]}\eta_{e[\rho'\sigma']}g^{\rho\rho'}g^{\sigma\sigma'}\Big\}\,. \end{aligned}\right\} \tag{21}$$

Here $(-\boldsymbol{G})$ is a scalar curvature density, and

$$h = \det(h_{k\mu}) = \sqrt{g} \qquad U_{\mu\nu} = U_{\nu,\mu} - U_{\mu,\nu}$$

$$V_{,\mu} = \partial V/\partial x^\mu \qquad \eta_{e[\rho\sigma]} = h_{e\rho,\sigma} - h_{e\sigma,\rho}$$

$4\pi\varepsilon^2 = \dfrac{8\pi f}{c^2}$ = Einstein's gravitational constant

f = Newton's gravitational constant

$\mu = \dfrac{\hbar k}{c}$ = the rest mass of a meson.

The equations of the field are given by variating I with regard to U_μ, V and $h_{i\mu}$ respectively (these field functions are considered to be mutually independent). That is;

$$\frac{\partial}{\partial x^{\mu}}(\mathfrak{U}^{\mu}-\mathfrak{V}^{\mu})=\frac{\partial \mathfrak{U}^{\mu}}{\partial x^{\mu}}-\frac{\partial}{\partial x^{\mu}}(hg^{\mu\nu}V_{,\nu})=0\,, \tag{22}$$

where

$$\mathfrak{U}^{\mu}=hU^{\mu} \quad \mathfrak{V}^{\mu}=hg^{\mu\nu}V_{,\nu}\,.$$

Hereafter Gothic letters express tensor (or vector) densities. The equations of U_μ are

$$\varkappa^2\mathfrak{U}^{\mu}+\frac{\partial \mathfrak{U}^{\mu\nu}}{\partial x^{\nu}}=\varkappa^2\mathfrak{V}^{\mu}. \tag{23}$$

Finally

$$\frac{\delta \mathfrak{G}}{\delta h_{i\mu}}=2\frac{\delta \mathfrak{R}}{\delta g_{\lambda\mu}}h_{i\lambda}=\left(\mathfrak{G}^{\lambda\mu}-\frac{1}{2}g^{\lambda\mu}\mathfrak{G}\right)h_{i\lambda}=\mathfrak{R}^{\lambda\mu}h_{i\lambda}$$

Here $\mathfrak{G}^{\lambda\mu}$ is Einstein's curvature tensor density. Further

$$\frac{\delta \mathfrak{M}}{\delta h_{i\mu}}=2\frac{\delta \mathfrak{M}}{\delta g_{\lambda\mu}}h_{i\lambda}=-\mathfrak{S}^{\lambda\mu}h_{i\lambda}\,, \tag{24}$$

where

$$\mathfrak{S}^{\lambda\mu}=-2\frac{\delta \mathfrak{M}}{\delta g_{\lambda\mu}}=h\{\varkappa^2(U^{\mu}-V^{\mu}_{,})(U^{\lambda}-V^{\lambda}_{,})+U^{\mu\nu}U^{\lambda}_{.\nu}\}-g^{\lambda\mu}\mathfrak{M} \tag{25}$$

Therefore the equations of $h_{i\mu}$ are

$$\mathfrak{R}^{\lambda\mu}=\mathfrak{G}^{\lambda\mu}-\frac{1}{2}g^{\lambda\mu}\mathfrak{G}=-8\pi\varkappa^2\mathfrak{S}^{\lambda\mu} \tag{26}$$

From these we know that $\mathfrak{S}^{\lambda\mu}$ is a symmetric energy momentum tensor density of the meson field. The relationship of this with the canonical one shall be given later.

iii). Transformation Group and Identities.

Now $I=\int \mathfrak{L}\,dx$ is invariant under the transformation groups of four types as follows;

a) the Lorentz group.

b) the guage transformation group, i. e.

$$\delta x^\mu=0 \quad \delta h_{i\mu}=0 \quad \delta V=\lambda(x) \quad \delta U_\mu=\lambda_{,\mu}\,. \tag{27}$$

Here λ is an arbitrary scalar function of x^μ.

Using the results of (i), we obtain the following relation corresponding to (9),

$$\frac{\delta \boldsymbol{L}}{\delta V}-\frac{\partial}{\partial x^\mu}\left(\frac{\delta \boldsymbol{L}}{\delta U_\mu}\right)=0 \tag{28}$$

This means that (22) can be derived from (23).

And corresponding to (10) and (11), we obtain

$$\frac{\partial \boldsymbol{L}}{\partial U_{\mu\nu}}+\frac{\partial \boldsymbol{L}}{\partial U_{\nu\mu}}\equiv 0 \tag{29}$$

and

$$\frac{\partial \boldsymbol{L}}{\partial U_\mu}+\frac{\partial \boldsymbol{L}}{\partial V_{,\mu}}\equiv 0 \tag{30}$$

respectively. (30) and (29) mean that the U_μ and its derivatives must be contained in $\boldsymbol{L}$ in the form $U_\mu-V_{1\mu}$, and $U_{\mu\nu}$ respectively. Finally corresponding to (12),

$$\frac{\partial}{\partial x_\mu}\left(\frac{\delta \boldsymbol{L}}{\delta U_\mu}\right)+\frac{\partial}{\partial x^\mu}\left(\frac{\partial \boldsymbol{L}}{\partial V_{,\mu}}\right)\equiv 0\,. \tag{31}$$

Combining (31) with (28), we obtain

$$\frac{\partial \boldsymbol{L}}{\partial V}\equiv 0 \tag{32}$$

This means V to be not contained in $\boldsymbol{L}$. On the contrary it can be shown that if (29), (30), and (32) are satisfied, $\boldsymbol{L}$ is invariant under the transformation (27).

c) Group of General Transformations of Coordinates.

This is defined by

$$\left.\begin{array}{c}\delta x^\mu=\xi^\mu(x) \quad \delta U_i=0 \quad \delta V=0 \\ \delta h_{i\mu}=-\dfrac{\partial \xi^\nu}{\partial x^\mu}h_{i\nu}\end{array}\right\} \tag{33}$$

For this group, it is convenient to rewrite (20) in the following form;

$$M = h\left\{\frac{\varkappa^2}{2}(U_i - V_{,i})(U_i - V_{,i}) + \frac{1}{4}U_{ik}U_{ik}\right\} \tag{20$'$}$$

Further in this case $\boldsymbol{R}$ is invariant as well as $\boldsymbol{G}$. Therefore instead of $\boldsymbol{L}$ we can take

$$L' = M - \frac{1}{8\pi\varepsilon^2}\boldsymbol{R} \tag{34}$$

as the Lagrangian of the system. Then as (9), we obtain

$$\frac{\delta M}{\delta V}V_{,\mu} + \frac{\delta M}{\delta U_i}U_{i,\mu} + \frac{\delta L'}{\delta h_{i\nu}}h_{i\nu,\mu} - \frac{\partial}{\partial x^\nu}\left(\frac{\delta L'}{\delta h_{i\nu}}h_{i\mu}\right) \equiv 0\,, \tag{35}$$

and as (10)

$$\frac{\partial \boldsymbol{R}}{\partial h_{i\nu,\rho}} + \frac{\partial \boldsymbol{R}}{\partial h_{i\rho,\nu}} \equiv 0\,. \tag{36}$$

That is, $\boldsymbol{R}$ contains the derivatives of h in the form $\eta_{i[\nu,\rho]} = h_{i\nu,\rho} - h_{i\rho,\nu}$. Conversely this is a proof for $\boldsymbol{R}$ to be invariant for the transformation (33). Further more putting the coefficients of each ξ^μ equal to zero, we obtain from (6).

$$\frac{\delta M}{\delta V}V_{,\mu} + \frac{\delta M}{\delta U_i}U_{i,\mu} + \frac{\delta L'}{\delta h_{i\nu}}h_{i\nu,\mu} + \frac{\partial}{\partial x^\rho}\left[\frac{\partial M}{\partial V_\rho}V_{,\mu} + \frac{\partial M}{\partial U_{i,\rho}}U_{i,\mu} + \frac{\partial L'}{\partial h_{i\nu,\rho}}h_{i\nu,\mu} - \delta^\rho_\mu L'\right] = 0\,. \tag{37}$$

Especially for the case where only the gravitational field exists,

$$\frac{\delta \boldsymbol{R}}{\delta h_{i\nu}}h_{i\nu,\mu} + \frac{\partial}{\partial x^\rho}\left[\frac{\partial \boldsymbol{R}}{\partial h_{i\nu,\rho}}h_{i\nu,\mu} - \delta^\rho_\mu \boldsymbol{R}\right] \equiv 0\,. \tag{37$'$}$$

And as (35), we obtain in this case

$$\frac{\partial \boldsymbol{R}}{\delta h_{i\nu}}h_{i\nu1\mu} - \frac{\partial}{\partial x^\rho}\left\{\frac{\delta \boldsymbol{R}}{\delta h_{i\rho}}h_{i\mu}\right\} \equiv 0\,. \tag{35$'$}$$

Subtracting (35)′ from (37)′,

$$\frac{\partial}{\partial x^\rho}\left\{\frac{\partial \boldsymbol{R}}{\partial h_{i\rho}}h_{i\mu}+\frac{\partial \boldsymbol{R}}{\partial h_{i\nu,\rho}}h_{i\nu,\mu}-\delta^\rho_\mu \boldsymbol{R}\right\}=0\,. \tag{38}$$

Using the field equation

$$\frac{\partial \boldsymbol{R}}{\partial h_{i\rho}}h_{i\mu}=-8\pi\varepsilon^2 \boldsymbol{S}^\rho_\mu$$

and putting

$$\boldsymbol{t}^\rho_\mu=-\frac{1}{8\pi\varepsilon^2}\left(\frac{\partial \boldsymbol{R}}{\partial h_{i\nu,\rho}}h_{i\nu,\mu}-\delta^\rho_\mu \boldsymbol{R}\right). \tag{39}$$

(38) becomes

$$\frac{\partial}{\partial x^\rho}\{\boldsymbol{S}^\rho_\mu+\boldsymbol{t}^\rho_\mu\}=0\,. \tag{38$'$}$$

This means that the energy momentum of the total system including the gravitational and the meson field is conserved. $\boldsymbol{t}^\mu_\nu$ is the energy momentum tensor density of the gravitational field, and is a tensor for the affin transformation but is not for the general transformation of coördinates. Strictly speaking it is a pseudo tensor density.

d) The transformation of the four legs.

This is given by

$$\left.\begin{array}{lll}\delta x^\mu=0 & \delta U_\mu=0 & \delta h_{i\mu}=\xi_{ik}h_{k\mu}\\ \delta V=0 & \delta g_{\mu\nu}=0 & \xi_{ik}=-\xi_{ki}\,,\end{array}\right\} \tag{40}$$

here ξ_{ik} is arbitrary functions of x^μ. For (40), $\boldsymbol{R}$ is not invariant but only $\boldsymbol{G}$. Here we do not discuss this transformation in detail.

Now as stated above I is invariant for the four groups, so according to the results of (i) several identities are obtained. Especially (28) and (35) are important, i. e. among 15–equations of fields, (these are (22) (23) (26)), there are five relations as the results of (28), (35). So the remaining 10–equations are independent.

Therefore the solutions of the field equations for V, U_μ, $h_{i\mu}$ contains 11–undetermined functions. But these functions can be compensated by λ of (27), ξ_μ of (33), ξ_{ik} of (40).

§ 2. Energy Momentum Tensor.

Now consider the relation between the symmetric energy momentum tensor density and the canonical one of any tensor or spinor field. At the first we assume the field functions Q_α to be spinor functions of any rank. Then Q_α are to be defined referring to the local systems $h^{(r)}_{i\mu}$ at all the world points.

To prove the above mentioned relation, it is useful to notice some natures of $h_{i\mu}$ and the convariant derivatives of any spinor functions.

The relation of $h_{i\mu}$'s at two neighbouring world points are given(15) by

$$h_{i\mu}(x+\Delta x)=h_{i\mu}(x)+\Gamma^{\lambda}_{\mu\nu}h_{i\lambda}\Delta x^{\nu}+\omega_{\nu,ki}h_{k\mu}\Delta x^{\nu}$$

$$h^{\mu}_{i}(x+\Delta x)=h^{\mu}_{i}(x)-\Gamma^{\mu}_{\lambda\nu}h^{\lambda}_{i}\Delta x^{\nu}+\omega_{\nu,ki}h^{\mu}_{k}\Delta x_{\nu}\,.$$

Here $\Gamma^{\lambda}_{\mu\nu}$ is the Christoffel's 3–index symbol and $\omega_{\nu,ki}$ is anti-symmetric with regard to $(k,\ i)$ in order to hold the orthogonality characters of $h_{i\mu}$ at the point $x+\Delta x$ as well as at x, and is defined by the above equations, i. e. by the following equations;

$$h_{i\mu,\nu}=\Gamma^{\lambda}_{\mu\nu}h_{i\lambda}-\omega_{\nu,ik}h_{k\mu}$$

And the convariant derivatives of Q_α are defined by

$$\nabla_\nu Q_\alpha=Q_{\alpha,\nu}-\frac{1}{2}D_{\alpha\beta,ik}Q_\beta\,\omega_{\nu,ik}\,,$$

where $D_{\alpha\beta,ik}$ is the transformation coefficient of Q for the Lorentz group. These are defined by

$$\left.\begin{array}{ll}\delta x^{\mu}=0 & \delta h_{i\mu}=\xi_{ik}h_{k\mu}\\ \partial Q_\alpha=\dfrac{1}{2}D_{\alpha\beta,ik}\xi_{ik}Q_\beta & \xi_{ik}(x)=-\xi_{ki}(x)\end{array}\right\}\qquad(41)$$

Now we consider a Langrangian $L(Q_\alpha,\ Q_{\alpha,\mu})$ which is only invariant for the Lorentz group. If we substitute $Q_{\alpha,\mu}$ with $\nabla_\mu Q_\alpha$, and introduce $h_{i\mu}$ appropriately and multiply $h=\det(h_{i\mu})$, we can get a new Lagrangian $\boldsymbol{L}$

---

(15) **K. Husimi:** cf. (13).
F. J. Belinfante: Physica 7 (1940) 305.

which is an invariant density for the general transformation of the coordinates (41) and that of the four legs $h_{i\mu}$. So for (41), we obtain from (9)

$$\frac{\delta \boldsymbol{L}}{\delta Q_\alpha}\left\{\frac{1}{2}(D_{\alpha\beta,ik}-D_{\alpha\beta,ki})\,Q_\beta\right\}+\frac{\delta \boldsymbol{L}}{\delta h_{i\mu}}h_{k\mu}-\frac{\delta \boldsymbol{L}}{\delta h_{k\mu}}h_{i\mu}=0\,.$$

Using the field equations $\delta\boldsymbol{L}/\delta Q_\alpha=0$ and the definition (24),

$$\boldsymbol{S}_{ik}-\boldsymbol{S}_{ki}=0 \tag{42}$$

is obtained, where

$$\boldsymbol{S}_{ik}=\boldsymbol{S}^{\lambda\mu}h_{i\lambda}h_{k\mu}\,.$$

Hereafter we define the symmetric energy momentum tensor density by

$$\boldsymbol{S}_{ik}=-\frac{\delta \boldsymbol{L}}{\delta h_{i\mu}}h_{k\mu}\,. \tag{43}$$

On the other hand the canonical one of the system is usually defined by

$$\boldsymbol{T}^\mu_\nu=\frac{\partial \boldsymbol{L}}{\partial Q_{\alpha,\mu}}Q_{\alpha,\nu}+\frac{\partial \boldsymbol{L}}{\partial h_{i\alpha,\mu}}h_{i\alpha,\nu}-\delta^\mu_\nu\boldsymbol{L}\,. \tag{44}$$

Further consider the general transformation of coordinates

$$\delta x^\mu=\xi^\mu \quad \delta Q_\alpha=0 \quad \delta h_{i\mu}=-\frac{\partial\xi^\nu}{\partial x^\mu}h_{i\nu}\,. \tag{45}$$

For (45), we can write (13), and (14) as follows using the definitions (43) and (44):

$$\boldsymbol{D}^\nu_\lambda=-\boldsymbol{T}^\nu_\lambda+\boldsymbol{S}^\nu_\lambda \tag{13$'$}$$

$$\boldsymbol{E}^{\nu\mu}_\lambda=-\boldsymbol{E}^{\mu\nu}_\lambda=\frac{\partial \boldsymbol{L}}{\partial h_{i\alpha,\nu}}(-h_{i\lambda}\delta^\mu_\alpha) \tag{14$'$}$$

and (11) becomes

$$\boldsymbol{S}^\nu_\lambda-\boldsymbol{T}^\nu_\lambda=\frac{1}{2}\frac{\partial}{\partial x^\mu}\left(\frac{\partial \boldsymbol{L}}{\partial h_{i\nu,\mu}}h_{i\lambda}-\frac{\partial \boldsymbol{L}}{\partial h_{i\mu,\nu}}h_{i\lambda}\right). \tag{11$'$}$$

Now

$$\frac{\partial \boldsymbol{L}}{\partial h_{j\beta,r}}=\frac{\partial \boldsymbol{L}}{\partial(\nabla_\nu Q_\alpha)}\frac{\partial(\nabla_\nu Q_\alpha)}{\partial h_{j\beta,r}}=-\frac{1}{2}D_{\alpha\varepsilon,ik}Q_\varepsilon\frac{\partial \boldsymbol{L}}{\partial Q_{\alpha,\nu}}\cdot\frac{\partial\omega_{\nu ik}}{\partial h_{j\beta,r}}$$

$$=\frac{1}{4} h^{\gamma}_{i} h^{\beta}_{k} \{ \boldsymbol{S}_{i[kj]}+\boldsymbol{S}_{k[ji]}-\boldsymbol{S}_{j[ik]} \}^{*}, \tag{46}$$

where

$$\boldsymbol{S}^{\rho}_{il}=-\boldsymbol{S}^{\rho}_{li}=-\frac{\partial \boldsymbol{L}}{\partial Q_{\alpha,\rho}} D_{\alpha\beta,il} Q_{\beta} \tag*{(47)$^{(16)}$}$$

Then from (11)′ (14)′ we obtain

$$\boldsymbol{S}^{\nu}_{\lambda}=\boldsymbol{T}^{\nu}_{\lambda}+\frac{1}{4}\frac{\partial}{\partial x^{\mu}}\left[h_{i\lambda} h^{\mu}_{j} h^{\nu}_{k} \{ \boldsymbol{S}_{j[ki]}+\boldsymbol{S}_{k[ij]}-\boldsymbol{S}_{i[jk]} \} \right] \tag{48}$$

$$\boldsymbol{E}^{\mu\nu}_{\lambda}=-\boldsymbol{E}^{\nu\mu}_{\lambda}=\frac{1}{4}\left[h_{i\lambda} h^{\mu}_{j} h^{\nu}_{k} \{ \boldsymbol{S}_{j[ki]}+\boldsymbol{S}_{k[ij]}-\boldsymbol{S}_{i[jk]} \} \right]. \tag*{(14)″}$$

Further we obtain the following equations by the same way as those of obtaining (37) provided that $\delta \boldsymbol{L}/\delta Q_{\alpha}=0$,

$$\frac{\partial \boldsymbol{T}^{\nu}_{\mu}}{\partial x^{\nu}}=-\frac{\delta \boldsymbol{L}}{\delta h_{i\nu}} h_{i\nu,\mu}.$$

And from (12)

$$\frac{\partial \boldsymbol{S}^{\nu}_{\mu}}{\partial x^{\nu}}=\frac{\partial \boldsymbol{T}^{\nu}_{\mu}}{\partial x^{\nu}}=-\frac{\delta \boldsymbol{L}}{\delta h_{i\nu}} h_{i\nu,\mu}. \tag*{(12)′}$$

Now if we neglect the gravitational effect by putting $h_{i\mu}=\delta_{i\mu}$ $\Gamma^{\lambda}_{\mu\nu}=0$ $\omega_{\rho,ik}=0$ and using the field equations, (in this case the distinctions between Latin and Greek indices and those between tensors snd tensor densities become unnecessary) we obtain the following equations from (42) (48) (12)

$$\left.\begin{aligned} S_{ik}=S_{ki}=T_{ki}+\frac{1}{4}\frac{\partial}{\partial x_{j}}\{ S_{j[ki]}+S_{k[ij]}-S_{i[jk]} \} \\ \frac{\partial S_{ki}}{\partial x^{k}}=\frac{\partial T_{ki}}{\partial x^{k}}=0, \end{aligned}\right\} \tag*{(48)′}$$

where

---

* In this paper the symmetric and antisymmetric parts of any tensor are to be written as follows;

$$A_{(\mu\nu)}=A_{\mu\nu}+A_{\nu\mu}$$
$$A_{[\mu\nu]}=A_{\mu\nu}-A_{\nu\mu}.$$

(16) K. Husimi: cf. (13).

$$\left.\begin{aligned} T_{ki}&=\frac{\partial L}{\partial Q_{\alpha,k}}Q_{\alpha,i}-\delta_{ik}L \\ L&=\text{a function of } Q_{\alpha}\,Q_{\alpha,\mu} \text{ only.}\end{aligned}\right\} \qquad (44)'$$

(48)′ is exactly the same as given by Prof. K. Husimi.(17) Further-more putting $\nu=4$ and integrating over all the three dimensional space $x^1\,x^2\,x^3$, provided that Q_α and their derivatives vanish so fast as not to contribute the integrals of the divergence-terms,

$$\int \boldsymbol{S}^4_\lambda dv=\overline{\boldsymbol{S}}^4_\lambda=\overline{\boldsymbol{T}}^4_\lambda+\frac{\partial}{\partial x^4}\int \boldsymbol{E}^{44}_\lambda dv=\overline{\boldsymbol{T}}^4_\lambda .$$

Thus the equivalence of the two energy momentum tensor densities has been proved.

The above argument also gives the proof for the case where Q_α is any tensor, if we take a spinor of a even rank as the field functions in the foregoing proof. But in the case of tensor field the treatment is more simple, so it is not vain to give a new proof for this case.

Now we assume Q_α to be a tensor, and the Lagrangian $L(Q_\alpha Q_{\alpha,\mu})$ being invariant only for the Lorentz group. If we introduce $g_{\mu\nu}$, and substitute $Q_{\alpha,\mu}=\partial Q_\alpha/\partial x^\mu$ with the convariant derivatives $\nabla_\mu Q_\alpha$ and multiply L with $\sqrt{g}=\sqrt{\det(g_{\mu\nu})}$, then we can get a new Lagrangian $\boldsymbol{L}=\sqrt{g}\cdot L(Q_\alpha, \nabla_\nu Q_\alpha, g_{\mu\nu})$ which is an invariant density for the general transformation group of coordinates. Hereafter we assume that Q_α is a convariant tensor, and its transformation character is given by

$$\delta x^\mu=\xi^\mu \quad \delta Q_\alpha=-C^{\beta\nu}_{\alpha\mu}\frac{\partial\xi^\mu}{\partial x^\nu}Q_\beta , \qquad (49)$$

where $C^{\beta\nu}_{\alpha\mu}$ is a constant tensor. Further the transformation of $g_{\mu\nu}$ is given by

$$\delta g_{\mu\nu}=-\frac{\partial\xi^\lambda}{\partial x^\mu}g_{\lambda\nu}-\frac{\partial\xi^\lambda}{\partial x^\nu}g^{\lambda\mu} .$$

Then the covariant derivative of Q_α is given by

$$\nabla_\mu Q_\alpha=\frac{\partial Q_\alpha}{\partial x^\mu}-\Gamma'^{\sigma}_{\mu\rho}\,C^{\beta\rho}_{\alpha\sigma}\,Q_\beta . \qquad (50)$$

---

(17) K. Husimi: cf. (13).

If we put $\xi^\mu = \varepsilon^\mu_\nu x^\nu (\varepsilon^\mu_\nu = -\varepsilon^\nu_\mu)$ in (49), the transformation (49) becomes the Lorentz transformation and

$$\delta Q_\alpha = -C^{\beta\nu}_{\alpha\mu}\,\varepsilon^\mu_\nu\,Q_\beta\,.$$

Comparing this with (41), we get

$$C^{\beta\nu}_{\alpha\mu} \to -\frac{1}{2}D_{\alpha\beta,\mu\nu}\,.$$

Now we define the symmetric energy momentum tensor by

$$\boldsymbol{S}_{\mu\nu} = -2\frac{\delta \boldsymbol{L}}{\delta g_{\mu\nu}} \tag{51}$$

and the canonical one by

$$\boldsymbol{T}^\mu_\nu = \frac{\partial \boldsymbol{L}}{\partial Q_{\alpha,\mu}}Q_{\alpha,\nu} + \frac{\partial \boldsymbol{L}}{\partial g_{\alpha\beta,\mu}}g_{\alpha\beta,\nu} - \delta^\mu_\nu \boldsymbol{L}\,. \tag{51'}$$

Then (13). (14) can be written for the transformation (49) using the field equations of Q,

$$\left.\begin{aligned} \boldsymbol{D}^\nu_\lambda &= \boldsymbol{S}^\nu_\lambda - \boldsymbol{T}^\nu_\lambda \\ \boldsymbol{E}^{\mu\nu}_\lambda = -\boldsymbol{E}^{\nu\mu}_\lambda &= -\left(\frac{\partial \boldsymbol{L}}{\partial g_{\alpha\nu,\mu}}g_{\alpha\lambda} - \frac{\partial \boldsymbol{L}}{\partial g_{\alpha\mu,\nu}}g_{\alpha\lambda}\right) \end{aligned}\right\} \tag{52}$$

and (11) becomes

$$\boldsymbol{S}^\nu_\lambda - \boldsymbol{T}^\nu_\lambda = \frac{\partial}{\partial x^\mu}\left\{g_{\alpha\lambda}\left(\frac{\partial \boldsymbol{L}}{\partial g_{\alpha\nu,\mu}} - \frac{\partial \boldsymbol{L}}{\partial g_{\alpha\mu,\nu}}\right)\right\} \tag{53}$$

From (53), of course, it is satisfied that

$$\frac{\partial \boldsymbol{S}^\nu_\lambda}{\partial x^\nu} = \frac{\partial \boldsymbol{T}^\nu_\lambda}{\partial x^\lambda}\,.$$

Further by exactly the same way as (12) we obtain

$$\frac{\partial \boldsymbol{S}^\nu_\lambda}{\partial x^\nu} = \frac{\partial \boldsymbol{T}^\nu_\lambda}{\partial x^\nu} = -\frac{\delta \boldsymbol{L}}{\delta g_{\alpha\beta}}g_{\alpha\beta,\lambda}\,. \tag{54}$$

Now we are to culculate (53) in order to compare this result with that given by Prof. K. Husimi. It is written that

$$\frac{\partial L}{\partial g_{\alpha\nu,\mu}} = -\frac{\partial L}{\partial Q_{\rho,\lambda}} C^{\beta\tau}_{\rho\sigma} \frac{\partial \Gamma^{\sigma}_{\lambda\tau}}{\partial g_{\alpha\nu,\mu}} Q_{\beta} .$$

Putting

$$\boldsymbol{S}^{\lambda\tau}_{\sigma} = \frac{\partial L}{\partial Q_{\rho,\lambda}} C^{\beta\tau}_{\rho\sigma} Q_{\beta} , \tag{55}$$

then

$$\boldsymbol{E}^{\mu\nu}_{\lambda} = -\frac{g_{\alpha\lambda}}{4} \{ \boldsymbol{S}^{(\mu[\nu)\alpha]} + \boldsymbol{S}^{(\nu[\alpha)\mu]} - \boldsymbol{S}^{(\alpha[\mu)\nu]} \} ,$$

where

$$\boldsymbol{S}^{(\mu[\nu)\alpha]} = \boldsymbol{S}^{(\mu\nu)\alpha} - \boldsymbol{S}^{(\mu\alpha)\nu} \quad \text{etc.}$$

Thus we obtain

$$\boldsymbol{S}^{\nu}_{\lambda} = \boldsymbol{T}^{\nu}_{\lambda} + \frac{1}{4} \frac{\partial}{\partial x^{\mu}} [\boldsymbol{S}^{(\mu[\nu)}{}_{\lambda]} + \boldsymbol{S}^{(\nu}{}_{[\lambda)}{}^{\mu]} - \boldsymbol{S}_{(\lambda}{}^{[\mu)\nu]}] . \tag{56}$$

If we put $g_{\mu\nu} = \delta_{\mu\nu}$, then we obtain by exactly the same way as the case of the spinor field,

$$\left. \begin{aligned} S_{\nu\lambda} = S_{\lambda\nu} = T_{\nu\lambda} + \frac{1}{4} \frac{\partial}{\partial x^{\mu}} \left[S_{(\mu[\nu)\lambda]} + S_{(\nu[\lambda)\mu]} - S_{(\lambda[\mu)\nu]} \right] . \\ \frac{\partial S_{\nu\lambda}}{\partial x^{\nu}} = \frac{\partial T_{\nu\lambda}}{\partial x^{\nu}} = 0 \end{aligned} \right\} \tag{57}$$

These results are completely identical with (48)′

§ 3. The weak Gravitational Field.

Now consider the linearization of the equations of the gravitational field in order to quantiz the field functions. For this purpose we assume that the gravitational field is so weak that it is able to put as follows ;

$$\left. \begin{aligned} h_{i\mu} &= \delta_{i\mu} + \varepsilon \theta_{i\mu}(x) \\ h^{\mu}_{i} &= \delta_{i\mu} - \varepsilon \theta_{i\mu}(x) . \end{aligned} \right\} \tag{58}$$

Then

$$\left. \begin{aligned} g_{\mu\nu} &= h_{i\mu} h_{i\nu} = \delta_{\mu\nu} + \varepsilon (\theta_{\mu\nu} + \theta_{\nu\mu}) \\ g^{\mu\nu} &= h^{\mu}_{i} h^{\nu}_{i} = \delta_{\mu\nu} - \varepsilon (\theta_{\mu\nu} + \theta_{\nu\mu}) \end{aligned} \right\} \tag{59}$$

$$h = 1 + \varepsilon \theta \quad \theta = \theta_{\mu\mu} \tag{60}$$

$$\eta_{l[\mu\nu]}=\varepsilon(\theta_{l\mu,\nu}-\theta_{l\nu,\mu})=\varepsilon\theta_{l[\mu,\nu]}\,. \tag{61}$$

Therefore $\boldsymbol{R}$ becomes as follows neglecting those terms of ε higher than the 2nd order,

$$\frac{1}{\varepsilon^2}\boldsymbol{R}=\theta_{\mu[\mu\nu]}\theta_{\rho[\rho\nu]}-\frac{1}{2}\theta_{\rho[\mu\nu]}\theta_{\mu[\rho\nu]}-\frac{1}{4}\theta_{\mu[\nu\rho]}\theta_{\mu[\nu\rho]}\,. \tag{62}$$

Further

$$\boldsymbol{M}=\frac{\varkappa^2}{2}(U_\mu-V_{,\mu})(U_\mu-V_{,\mu})+\frac{1}{4}U_{\mu\nu}U_{\mu\nu}-\frac{\varepsilon}{2}S_{\mu\nu}T_{\mu\nu} \tag{63}$$

$$\begin{aligned}T_{\mu\nu}=\varkappa^2(U_\mu-V_{,\mu})(U_\nu-V_{,\nu})+U_{\mu\alpha}U_{\nu\alpha}\\-\delta_{\mu\nu}\left\{\frac{\varkappa^2}{2}(U_\rho-V_{,\rho})(U_\rho-V_{,\rho})+\frac{1}{4}U_{\rho\lambda}U_{\rho\lambda}\right\},\end{aligned} \tag{64}$$

where

$$s_{\mu\nu}=\theta_{\mu\nu}+\theta_{\nu\mu}\,.$$

Now (62), (63) are approximate equations, but hereafter let us consider these equations to be exact, that is, consider a system which is described by the Lagrangian (62) and (63). And as the field functions we take $\theta_{i\mu}$, U_μ and V, which are defined in the Minkowskie space ($x^\mu=x, y, z, ict$). Therefore it is unnecessary hereafter to distinguish the Latin and Greek letters, or the upper and lower indices, and tensors and tensor densities.

i) The Equations of Fields.

As the Lagrangian of the system we take

$$L=M-\frac{1}{8\pi}\left\{\theta_{\mu[\mu\sigma]}\theta_{\nu[\nu\sigma]}-\frac{1}{2}\theta_{\mu[\nu\sigma]}\theta_{\nu[\mu\sigma]}-\frac{1}{4}\theta_{\mu[\nu\sigma]}\theta_{\mu[\nu\sigma]}\right\}. \tag{65}$$

Instead of this, we can take

$$\begin{aligned}L'=L-\frac{1}{4\pi}\frac{\partial}{\partial x^\nu}\Big[\frac{1}{\varepsilon}\frac{\partial}{\partial x^\sigma}(\delta\theta_{\nu\sigma}-\theta_{\nu\sigma})\\-\frac{\partial}{\partial x^\sigma}(\theta\cdot\vartheta_{\nu\sigma})-\theta_{l\nu}\frac{\partial}{\partial x^\sigma}(\delta_{l\sigma}\theta-\theta_{l\sigma})\Big].\end{aligned} \tag{65$'$}$$

But we get the same equations of fields from (65)′ as those from (65), so we take (65) as the Lagrangian in this paper, the reason for this will be given later.

Now we put as follows

$$\left.\begin{array}{c} s_{\mu\nu}=\theta_{\mu\nu}+\theta_{\nu\mu} \qquad a_{\mu\nu}=\theta_{\mu\nu}-\theta_{\nu\mu} \\ \theta_{\mu\nu}=\frac{1}{2}(s_{\mu\nu}+a_{\mu\nu}), \end{array}\right\} \tag{66}$$

then R can be written as follows

$$\frac{1}{\varepsilon^2}R=\frac{1}{4}\left[s_{\mu[\mu\sigma]}s_{\nu[\nu\sigma]}-\frac{1}{2}s_{\mu[\nu\sigma]}s_{\nu[\mu\sigma]}-\frac{1}{4}s_{\mu[\nu\sigma]}s_{\mu[\nu\sigma]}\right]$$

$$-\frac{1}{2}\frac{\partial}{\partial x^\rho}\Big[s_{\nu\nu}s_{\mu\rho,\mu}+s_{\rho\nu,\mu}a_{\mu\nu}-s_{\mu\nu,\mu}a_{\rho\nu}$$

$$+\frac{1}{2}(a_{\rho\nu,\mu}a_{\mu\nu}-a_{\mu\nu,\mu}a_{\rho\nu})\Big]. \tag{62$'$}$$

Therefore in variating L with regard to $\theta_{\mu\nu}$ we get only the equations of $s_{\mu\nu}$, and $a_{\mu\nu}$ does not appear in these equations. So $a_{\mu\nu}$ is completely undetermined. This fact is already stated in page 10. Therefore we put $a_{\mu\nu}$ out of the question and restrict the problem in the treatment of only.

Now consider the transformation of four legs corresponding (40)

$$\delta x^\mu=0 \qquad \delta\theta_{i\mu}=\xi_{i\mu},$$

neglecting the terms of ε higher than the 1st order. Then because of the antisymmetry of $\xi_{\mu\nu}$, it becomes

$$\delta s_{\mu\nu}=\xi_{\mu\nu}+\xi_{\nu\mu}=0$$

$$\delta a_{\mu\nu}=\xi_{\mu\nu}\xi_{\nu\mu}=2\xi_{\mu\nu}.$$

Thus in the approximation mentioned above only $a_{\mu\nu}$'s are varied but $s_{\mu\nu}$ are invariant and so it becomes unnecessary to consider this transformation restricting the problem within the behaviour of $s_{\mu\nu}$ only.

Though only G is invariant for this transformation, it is sufficient to use R instead of G as the Lagrangian of the gravitational field on account of the fact just mentioned above. This is the reason for taking L as the Lagrangian of the total system instead of L'.

Further G is written as follows, using $g_{\mu\nu}$ instead of $h_{i\mu}$,

$$G=-R'+\frac{\partial}{\partial x^{\mu}}\left(g^{\lambda\nu}\frac{\partial R'}{\partial g^{\lambda\nu}_{\mu}}\right)$$

$$R'=\sqrt{g}\,g^{\mu\nu}(\Gamma^{\beta}_{\mu\alpha}\Gamma^{\alpha}_{\nu\beta}-\Gamma^{\alpha}_{\mu\nu}\Gamma^{\beta}_{\alpha\beta})\,. \tag{67}$$

Now using (59), it is written

$$g_{\mu\nu}=\delta_{\mu\nu}+\varepsilon s_{\mu\nu}\,,$$

so in the approximation mentioned above,

$$\frac{1}{\varepsilon^2}R'=\frac{1}{2}s_{\rho\nu,\mu}s_{\rho\mu,\nu}-\frac{1}{4}s_{\mu\nu,\rho}s_{\mu\nu,\rho}-\frac{1}{2}s_{\rho\rho,\nu}s_{\mu\nu,\mu}+\frac{1}{4}s_{\rho\rho,\nu}s_{\mu\mu,\nu} \tag{68}$$

can be obtained. On the other hand (62) can be rewritten as follows,

$$\frac{1}{\varepsilon^2}R=\frac{1}{2}s_{\rho\nu,\mu}s_{\rho\mu,\nu}-\frac{1}{4}s_{\mu\nu,\rho}s_{\mu\nu,\rho}-\frac{1}{2}s_{\rho\rho,\nu}s_{\mu\nu,\mu}+\frac{1}{4}s_{\rho\rho,\nu}s_{\mu\mu,\nu}$$
$$-\frac{1}{2}\frac{\partial}{\partial x^{\rho}}\Big[s_{\nu\nu}s_{\mu\rho,\mu}+s_{\rho\nu,\mu}a_{\mu\nu}-s_{\mu\nu,\mu}a_{\rho\nu}$$
$$+\frac{1}{2}(a_{\rho\nu,\mu}a_{\mu\nu}-a_{\mu\nu,\mu}a_{\rho\nu})-\frac{1}{2}(s_{\rho\nu}s_{\mu\nu,\mu}-s_{\mu\nu}s_{\rho\nu,\mu})\Big]\,. \tag{69}$$

Thus R differs from R' by a divergence term. Consider the transformation corresponding to (33), that is, (here the order of the approximation above stated is to be noticed),

$$\delta x^{\mu}=\varepsilon\xi^{\mu}\qquad \delta h_{i\mu}=-\varepsilon\frac{\partial\xi^{i}}{\partial x^{\mu}}\,,$$

or

$$\delta\theta_{\mu\nu}=-\frac{\partial\xi_{\mu}}{\partial x^{\nu}}\qquad \delta s_{\mu\nu}=-\left(\frac{\partial\xi_{\mu}}{\partial x^{\nu}}+\frac{\partial\xi_{\nu}}{\partial x^{\mu}}\right)$$

$$\delta a_{\mu\nu}=-\left(\frac{\partial\xi_{\mu}}{\partial x^{\nu}}-\frac{\partial\xi_{\nu}}{\partial x^{\mu}}\right).$$

For this (69) is invariant but (68) is not. Therefore we must use R instead of R'.

Now the equations of field are written as follows;

$$\left.\begin{array}{c}\dfrac{\partial U_\mu}{\partial x^\mu}-\Box V=\varepsilon\dfrac{\partial}{\partial x^\mu}\left\{\dfrac{s}{2}(V_{,\mu}-U_\mu)-s_{\mu\nu}(V_{,\nu}-U_\nu)\right\}\\ s=s_{\mu\mu}\,,\end{array}\right\}\tag{70}$$

where

$$\kappa^2\left\{(U_\mu-V_{,\mu})-\varepsilon s_{\mu\nu}(U_\nu-V_{,\nu})\right\}+\frac{\varepsilon}{2}s\kappa^2(U_\mu-V_{,\mu})$$
$$+\frac{\partial}{\partial x^\nu}\left\{\left(1+\frac{\varepsilon}{2}s\right)U_{\mu\nu}-\varepsilon(\delta_{\mu\lambda}U_{\lambda\nu}-s_{\nu\lambda}U_{\lambda\mu})\right\}=0\,.\tag{71}$$

For $a_{\mu\nu}$ we obtain only the following trivial equation

$$0=0\,.$$

That is, $a_{\mu\nu}$ is completely undetermined but this indefiniteness can be eliminated by the transformation of the four legs. For $s_{\mu\nu}$, we get the following,

$$\Box s_{\mu\nu}+\frac{\partial^2 s}{\partial x^\mu\partial x^\nu}-\frac{\partial^2 s_{\mu\lambda}}{\partial x^\nu\partial x^\lambda}-\frac{\partial^2 s_{\nu\lambda}}{\partial x^\mu\partial x^\lambda}+\delta_{\mu\nu}\left(\frac{\partial^2 s_{\alpha\beta}}{\partial x^\alpha\partial x^\beta}-\Box s\right)$$
$$=-8\pi\varepsilon T_{\mu\nu}\tag{72}$$

where $T_{\mu\nu}$ is given by (63).

Putting $\mu=4$ $\nu=k=1.2.3$ in (72) we obtain the following, (after here the Latin and the Greek index represent 1. 2. 3. and 1. 2. 3. 4. respectively.)

$$\Delta s_{4k}-\frac{\partial^2 s_{kl}}{\partial x^4\partial x^e}+\frac{\partial^2 s_{ll}}{\partial x^4\partial x^k}-\frac{\partial^2 s_{4l}}{\partial x^k\partial x^l}=-8\pi\varepsilon T_{4k}\,.\tag{72·1}$$

Further for $\mu=\nu=4$,

$$\frac{\partial^2 s_{kl}}{\partial x^k\partial x^l}-\Delta s_{kk}=-8\pi\varepsilon T_{44}\,.\tag{72·2}$$

In (72), if $\mu\neq4$, $\nu\neq4$, then these are the differential equations of the 2nd order with respect to the time, whereas (72·1) (72·2) are the 1st and 0–th order. So we can consider the latter to be initial conditions for the former.

In the end of this paragraph we give the following by making the four ensional divergence of (72), and taking notice for the left side being zero ;

$$0=\frac{\partial T_{\mu\nu}}{\partial x^{\mu}}\,. \tag{73}$$

ii) Group of the Transformation.

Now it is already stated in § 1 (iii) that L is invariant for the following transformation groups.

a) Guage transformation ;

$$\delta x^{\mu}=0 \quad \delta s_{\mu\nu}=\delta a_{\mu\nu}=0 \quad \delta V=\lambda \quad \delta U_{\mu}=\lambda_{,\mu}\,. \tag{27$'$}$$

On account of this there is a dependence between (70) and (71), and so an arbitrary function exisits in the solution of the U, V-field equations. But we can put $V=0$ by (27)$'$. Then (70) becomes

$$\frac{\partial U_{\mu}}{\partial x^{\mu}}=\varepsilon\frac{\partial}{\partial x^{\mu}}\left[s_{\mu\nu}U_{\nu}-\frac{s}{2}U_{\mu}\right], \tag{74}$$

and (71) becomes

$$\kappa^{2}\left\{U_{\mu}-\varepsilon s_{\mu\nu}U_{\nu}+\frac{\varepsilon}{2}sU_{\mu}\right\}$$
$$+\frac{\partial}{\partial x^{\nu}}\left\{\left(1+\frac{\varepsilon}{2}s\right)U_{\mu\nu}-\varepsilon(s_{\mu\lambda}U_{\lambda\nu}-s_{\nu\lambda}U_{\lambda\mu})\right\}=0\,. \tag{75}$$

(74) can be yet derived from (75), and we can interprete (74) as a condition for the field equations (75), because (74) contains only the 1st derivative with respect to the time. And this condition (74) is always satisfied owing to (75). Thus the number of independent components of the field functions decreases by unity.

b) The general transformation of coordinates.

$$\delta x^{\mu}=\varepsilon\xi_{\mu} \qquad \delta s_{\mu\nu}=-\left(\frac{\partial\xi_{\mu}}{\partial x^{\nu}}+\frac{\partial\xi_{\nu}}{\partial x^{\mu}}\right)$$

$$\delta a_{\mu\nu}=-\left(\frac{\partial\xi_{\mu}}{\partial x^{\nu}}-\frac{\partial\xi_{\nu}}{\partial x^{\mu}}\right) \quad \delta U_{\mu}=-\varepsilon U_{\nu}\frac{\partial\xi_{\nu}}{\partial x^{\mu}}, \tag{76}$$

where we reserve the 1st order of ε for U, the 0–th order for $s_{\mu\nu}$ and $a_{\mu\nu}$. For (76), L is an invariant density in the approximation mentioned above. From this invariancy, we obtain the following relation corresponding to (35),

$$\frac{\partial}{\partial x^{\mu}}\left(\frac{\delta L}{\delta s_{\mu\nu}}\right)=0\,, \tag{77}$$

where the equations of U are used. This is already stated in (73). (77) means that ten equations (72) are not mutnally independent but four equations among them can be derived from the rest. Therefore four arbitrary functions exist in the solution of the equations of $s_{\mu\nu}$. Therefore we remove this difficulty by the following way.(18)

Putting

$$\gamma_{\mu\nu}=s_{\mu\nu}-\frac{1}{2}\delta_{\mu\nu}s\,, \tag{78}$$

(72) can be rewritten using $\gamma_{\mu\nu}$ instead of $S_{\mu\nu}$

$$\Box\gamma_{\mu\nu}-\frac{\partial^2\gamma_{\mu\rho}}{\partial x^{\nu}\partial x^{\rho}}-\frac{\partial^2\gamma_{\nu\rho}}{\partial x^{\mu}\partial x^{\rho}}+\delta_{\mu\nu}\frac{\partial^2\gamma_{\rho\sigma}}{\partial x^{\rho}\partial x^{\sigma}}=-8\pi\varepsilon\,T_{\mu\nu}\,. \tag{72$'$}$$

For (76), $\gamma_{\mu\nu}$ is transformed by

$$\delta\gamma_{\mu\nu}=-\frac{\partial\xi_{\mu}}{\partial x^{\nu}}-\frac{\partial\xi_{\nu}}{\partial x^{\mu}}+\delta_{\mu\nu}\frac{\partial\xi_{\rho}}{\partial x^{\rho}}\,.$$

Therefore

$$\delta\frac{\partial\gamma_{\mu\nu}}{\partial x^{\mu}}=\frac{\partial}{\partial x^{\mu}}(\delta\gamma_{\mu\nu})-\gamma_{\mu\nu,\rho}\frac{\partial\delta x^{\rho}}{\partial x^{\mu}}=-\Box\xi_{\nu}-\varepsilon\gamma_{\mu\nu,\rho}\frac{\partial\xi_{\rho}}{\partial x^{\mu}}\,.$$

Now we take ξ_{μ} so as to satisfy

$$\frac{\partial\gamma_{\mu\nu}}{\partial x^{\nu}}=\Box\xi_{\mu}\,,$$

then transforming $\partial\gamma_{\mu\nu}/\partial x^{\nu}$ by (76) with above ξ_{μ} we get

$$\frac{\partial\gamma'_{\mu\nu}}{\partial x'^{\nu}}=\frac{\partial\gamma_{\mu\nu}}{\partial x^{\nu}}+\delta\left(\frac{\partial\gamma_{\mu\nu}}{\partial x^{\nu}}\right)=0\,,$$

where the transformed quantities are expressed by a prime. Thus we can put

$$\frac{\partial\gamma_{\mu\nu}}{\partial x^{\nu}}=0$$

---

(18) A. Einstein: cf. (7).

by using an appropriate coordinate system as shown by Einstein. In this case (72)′ can be reduced to

$$\Box\gamma_{\mu\nu}=-8\pi\varepsilon T_{\mu\nu}. \tag{80}$$

But here we can yet transform all the equations covariantly by (76) using the following ξ_μ

$$\Box\xi_\mu=0 \qquad \partial\xi_\mu/\partial x^\mu=0, \tag{81}$$

provided that the above mentioned approximation is noticed. For in this transformation

$$\delta\frac{\partial\gamma_{\mu\nu}}{\partial x^\nu}\cong\frac{\partial}{\partial x^\nu}\delta\gamma_{\mu\nu}=0$$

and

$$\delta(\Box\gamma_{\mu\nu}+8\pi\varepsilon T_{\mu\nu})=-\Box\left(\frac{\partial\xi_\mu}{\partial x^\nu}+\frac{\partial\xi_\nu}{\partial x^\mu}\right)+O(\varepsilon^2)=0.$$

In the last we must investigate whether the condition (79) contradicts with the field equation (80) or not. Taking the divergence of (80) with respect to x^ν we obtain

$$\Box\frac{\partial\gamma_{\mu\nu}}{\partial x^\nu}=-8\pi\varepsilon\frac{\partial T_{\mu\nu}}{\partial x^\nu}.$$

Because of the vanishing of the right-hand side it becomes

$$\Box\frac{\partial\gamma_{\mu\nu}}{\partial x^\nu}=0. \tag{82}$$

So if we assume the following conditions at any instant (for example $x^4=0$)

$$\frac{\partial\gamma_{\mu\nu}}{\partial x^\nu}=0 \qquad \frac{\partial}{\partial x^4}\left(\frac{\partial\gamma_{\mu\nu}}{\partial x^\nu}\right)=0 \tag{83}$$

then on account of (82), the condition (79) is always satisfied.

Thus the four arbitrary functions mentioned above are restricted so as to satisfy the eight conditions (83). Therefore two components among the field functions $s_{\mu\nu}$ are really independent.

Following Fierz[19] the transformation

$$\left.\begin{aligned} \delta S_{\mu\nu} &= -\left(\frac{\partial \hat{\xi}_\mu}{\partial x^\nu} + \frac{\partial \hat{\xi}_\nu}{\partial x^\mu}\right) \\ \Box \hat{\xi}_\mu = 0 &\quad \frac{\partial \hat{\xi}_\mu}{\partial x^\mu} = 0 \end{aligned}\right\} \tag{84}$$

is called as the guage transformation of the 2nd kind.

The results above stated are, in sum, as follows that the meson field has three independent components whereas the gravitational field has only two. Following the results given by Dirac, Fierz, Pauli[20] we know that the former represents particles of the spin 1 and the rest mass $\mu = \hbar\varkappa/c$, on the other hand the latter represents particles of the spin 2, and the rest mass 0, provided that the both fields are quantized with Heisenberg-Pauli's method, which will be discussed in (II). But the above arguments are only correct within the already stated approximation with regard to the interaction constant ε and only in the case of $\varepsilon = 0$, these are exactly correct, and (84) becomes the guage transformation of the 2nd kind in the strict sense. On the contrary if $\varepsilon \neq 0$, then we must consider besides the transformation of $s_{\mu\nu}$, those of U–field and of the coordinate system together. Therefore, strictly speaking, (84) is not the guage transformation in the sense given by Fierz.

In the end of this paragraph we give a few remarks that if we put $\varkappa = 0$, we obtain the equations for photons interacting with gravitons which were given by Rosenfeld.[21] But in his paper, R' is used instead of R, so this Lagrangian is not a guage invariant density in the sense of Fierz. Therefore the energy momentum tensor is also not invariant.

On the other hand if we put $U_\mu = V = 0$, then we obtain the same results as given by Pauli,[21] but he also used R', so his Lagrangian is not invariant.

Department of Physics,
Osaka Imperial University.

---

(19) M. Fierz: cf. (8).

(20) L. Rosenfeld: cf. (5).

(21) W. Pauli: cf. (9).

Progress of Theoretical Physics Vol. II, No. 2, Mar.~Jun., 1947.

On the Effect of the Field Reactions on the Interaction of Mesotrons and Nuclear Particles. IV.*

By Shin-ichirô TOMONAGA.

(Received July 8, 1946)

§ 4. Phisical consequences.

We shall now proceed to the discussion of the physical consequences obtained by our calculations. It is true that our calculations must be more refined and generalized before we can draw any decisive conclusions, because they are based on a too much simplified model of the system (the assumption that only longitudinal mesotrons interact with the nuclear particle which is thought infinitely heavy etc.). But some remarkable results obtained here seems to have bearing on experimental evidences to which the ordinary perturbation theory could not give any satisfactory explanations.

A :—Scattering of a mesotron by a nuclear particle.

As mentioned before, the perturbation theory gives the cross-section for the scattering of a mesotron on colliding with a nuclear particle

$$\sigma=\frac{4\pi}{k^2}\frac{(4\pi^2 l^2 g(k)^2 K k)^2}{K^2}\,, \tag{4·1}$$

where k and K represent respectively the momentum and the energy of the incident mesotron. Inserting $g(k)$ of $(1 \cdot 11)$ we obtain

$$\sigma=64\pi^5 l^4\frac{k^4 F(k)^4}{K^2}\,. \tag{4·2}$$

This σ depends on l, which is determined usually in the following way: we determine first $\varkappa$ by measuring the mass of the mesotron, which gives rise to $\varkappa=1/1.7\times 10^{+13}\,\mathrm{cm}^{-1}$. The coupling constant g is then determined by the experimental data for the nuclear forces: $g_1^2/\hbar c \simeq 0.1$. These values of $\varkappa$ and g give then $l \simeq 10^{-14}$ cm. Further it seems probable that the cut-off momentum $\boldsymbol{K}$ is of the order of the rest mass of the nuclear particle, thus $\boldsymbol{K} \simeq 10\,\varkappa$.

---

* The sections up to § 3 are contained in Part I (Progr. Theor. Phys. 1 (1946), 83), Part II (I (1946), 109) and Part III (II (1947), 6) of this paper.

If we adopt this value for l, the cross-section of a fast mesotron with an energy of the order of $\boldsymbol{K}$ being scattered by a nuclear particle is given by

$$\sigma \simeq 64\pi^5 l^4 \boldsymbol{K}^2 \simeq 3.6 \times 10^{-25}\ \text{cm}^2, \tag{4·3}$$

provided that the incident particle has an energy not lying in the cut-off region. Further the cross-section of a slow mesotron with an energy of the order of $\varkappa$ is

$$\sigma \simeq 32\pi^5 l^4 \varkappa^2 \simeq 8 \times 10^{-27}\ \text{cm}^2. \tag{4·4}$$

These results are now to be compared with the experimental results obtained by Wilson⁽¹¹⁾ who has found considerablly smaller cross-sections

$$\sigma \lesssim 10^{-28}\ \text{cm}^2 \tag{4·5}$$

for fast mesotrons with energies of the order of 10^9 eV and

$$\sigma \lesssim 10^{-27}\ \text{cm}^2 \tag{4·6}$$

for slow mesotrons with energies of the order of 10^8 eV. Thus the theoretical results (4·3) and (4·4) are obviously too large.

Now, our result, which is calculated taking the inertial as well as the damping reactions into account, is

$$\begin{aligned}\sigma_+ = \sigma_- &= \frac{1}{2}\,\frac{\pi}{k^2}\,\frac{(4\pi^2 g(k)^2 K k)^2}{\boldsymbol{K}_3^2 K^4 + \frac{1}{4}(4\pi^2 g(k)^2 K k)^2} \\ &= \frac{8\pi^5 k^4}{\boldsymbol{K}_3^2 K^4 + 4\pi^4 k^8}\,.\end{aligned} \tag{4·7}$$

This gives, for slow mesotrons with the energies of the order of $\varkappa$

$$\sigma = \frac{8\pi^5}{\boldsymbol{K}_3^2}\left(\frac{k}{K}\right)^4 \simeq 9 \times \frac{1}{\boldsymbol{K}^2} \simeq 2.6 \times 10^{-27}\ \text{cm}^2. \tag{4·8}$$

This value is smaller than (4·4) and is more satisfactory. When the energy

---

(11) J. G. Wilson: Proc. Roy. Soc., 147 (1940), 73.

of the incident mesotron is so large as of the order of $\boldsymbol{K}$, our result is inaccurate because we have neglected higher powers of $K/\boldsymbol{K}$ in our calculation. But our result would still give a correct order of magnitude:

$$\sigma \cong \frac{4.8}{\boldsymbol{K}^2} \cong 1.3 \times 10^{-27}\,\text{cm}^2. \tag{4·9}$$

This is much smaller than (4·3) so that it is more satisfactory than the result of the perturbation theory. The values (4·8) and (4·9) could be still reduced by assuming a still larger $\boldsymbol{K}$. But when $\boldsymbol{K}$ is much larger than the rest mass of the nuclear particle, our results based on the assumption of infinitely heavy nuclear particle becomes inaccurate, because in such a case the recoil of the nuclear particle can no more be neglected.

In this way it seems possible that the smallness of the scattering cross-section is accounted for as due to the large inertial reaction of the self-field.

If this explanation for the smallness of the scattering cross-section is correct, there is a possibility of determining the cut-off momentum independently of the value of l, since our formula (4·7) does not contain l. Then we shall find that our assumption of $\boldsymbol{K} \cong 10\,\varkappa$ is a reasonable one.

B:—Magnitude of nuclear forces and their range.

Either of the formulas (1·26), (2·35) or (3·38) giving the self-energy of the nuclear particle in case of strong coupling has the principal term

$$E = -\frac{1}{2} l^2 \boldsymbol{K}_1^3, \tag{4·10}$$

while according to (3·52), which is the result of the perturbation theory, we have

$$E = -l^2 \boldsymbol{K}_1^3. \tag{4·11}$$

One notes that (4·10) is only a half of (4·11). As mentioned in § 1B this fact implies that the perturbation theory gives the erroneous result for the intensity of the self-field, the result which is too large by the factor $\sqrt{2}$, if this procedure is applied to the case of strong coupling without respect the divergence of this process in such a case.

This fact results in that, if the coupling is strong, the perturbation theory gives too large values of the nuclear forces and that, if one determines the coupling constant g from the experimental data by using the perturbation theory, one obtains too small a value for g.

It is usually assumed that $g^2/\hbar c \simeq 0.1$ and $K \doteq 10\,\varkappa$. Using these values, the parameter V defined by $(3\cdot40)$ is found:

$$V^2 = l^2 K_2^2 = \frac{1}{4\pi^2}\,\frac{g^2}{\hbar c}\,\frac{1}{\varkappa^2}\,\frac{4\pi}{2}\,K^2 \doteq 1.6. \qquad (4\cdot12)$$

Comparing our result obtained in § 3D (see Fig. 4 and 5), this value of V is found not to be sufficiently small to allow the application of the perturbation theory. Thus, for the determination of the coupling constant the use of the perturbation theory is not allowed and the true value of g must be larger than that usually believed.

We next notice that if the coupling is strong, two parameters α and β introduced in $(3\cdot6)$ have non-vanishing values. This fact results in that the form of the nuclear potential will have different form from that ordinarily assumed, since this ordinary form $e^{-\varkappa r}/r$ corresponds to the vanishing values of α and β (provided that no cut-off process is introduced).

According to $(3\cdot37)$ we obtain for $m=1/2$

$$\left.\begin{aligned} \alpha &= \frac{1}{l^2 K_3}\left(\frac{1}{2}-\frac{4}{3}\right) = -\frac{5}{6}\,\frac{1}{l^2 K_3} \\ \beta &= \frac{-1}{l^2 K_3}\left(\frac{1}{2}+\frac{4}{3}\right) = -\frac{11}{6}\,\frac{1}{l^2 K_3}, \end{aligned}\right\} \qquad (4\cdot13)$$

this gives negative values for both α and β. The existence of these negative parameters in the denominators of $(3\cdot6)$ results in that the range of the nuclear forces is considerabily smaller than that given by the nuclear potential of the ordinary form $e^{-\varkappa r}/r$. We can thus conclude that, if the coupling is strong, the range of the nuclear forces is smaller than $1/\varkappa$.

If one adopt for the mesotron mass the value 200 times larger than the mass of the electron one obtain $1/\varkappa \simeq 1.7\times10^{-13}$ cm, which is usually believed to be the range of the nuclear forces. But the situation mentioned above makes the range smaller even if the mass of the mesotron is 200 electron mass.

Now, Hoisington, Share and Breit⁽¹²⁾ have analized the experimental data for the proton-proton scattering and found that the range of the proton-

---

(12) L. E. Hoisington, S. S. Share and G. Breit: P. R., **56** (1938), 884.

proton force is in fact smaller than the above value of $1/\kappa$. They have found that in order to obtain the range of the nuclear forces so as to fit with the scattering data, the mass of the mesotron had to be 326 times the electron mass instead of 200 times. But this large mass seems to be very difficult to be reconciled with the mass-determination experiments, Our result mentioned above seems just remove this difficulty.

If the explanation on this line of the result of Hoisington and others is correct, α and β can be determined by analizing the form of the nuclear potential. Although in the present stage of our theory, an accurate determination of α and β can not be carried out, the comparison of our result with the result of Hoisington and others shows that these parameters are at least of the order of κ. Thus by putting $\alpha \cong \kappa$ in $(4 \cdot 13)$:

$$\frac{1}{l^2 \boldsymbol{K}_3} \cong \kappa\,, \qquad (4 \cdot 14)$$

we find

$$V^2 \cong \frac{1}{2}\left(\frac{\boldsymbol{K}}{\kappa}\right). \qquad (4 \cdot 15)$$

Our assumption $\boldsymbol{K} = 10\,\kappa$, which is probable from the smallness of the scattering cross-section, gives then

$$V^2 \cong 5\,. \qquad (4 \cdot 16)$$

Inserting this V into $(4 \cdot 12)$, we obtain

$$g^2/\hbar c \cong 0.3\,. \qquad (4 \cdot 17)$$

This value of g is considerablly larger than the value 0.1 usually assumed, but it will be by no means too large, since the usual value $g=0.1$ is certainly too small owing to the error of the perturbation theory. Moreover, this value 0.1 has been derived by assuming that the nuclear force has the range of $1/\kappa$. If we use here the value found by Hoisington and others an appreciablly larger g would be obtained even if we disregard the error of the perturbation theory.

C :—Existence of stable or meta-stable proton-isobars.

When $(4 \cdot 14)$ holds, $(3 \cdot 38)$ gives rise to the interval between rotational levels

$$\Delta E \cong \kappa\left(m-\frac{1}{2}\right). \tag{4·18}$$

This makes probable that the state with $m=3/2$ is stable. Even if this state is not stable, this state will be meta-stable and will have very long life, because we obtain from (3·37)

$$a \cong \kappa\left(\frac{3}{2}-\frac{4}{3}\right)=\frac{1}{6}\kappa\,, \tag{4·19}$$

which is far smaller than κ, and hence the wave function of this state can be approximated very well by the wave function not extending to infinity.

The state with $m=5/2$ will be certainly no more stable. But even for this state we have

$$a \cong \frac{7}{6}\kappa \cong \kappa\,, \tag{4·20}$$

so that a does not much exceed κ. Thus this state will have a life sufficiently long to play a rôle in collision processes as long life intermediate state.

According to our theory, it is thus possible that the proton-isobar with charge 2 or −1 will be observed by experiment.

There are two Wilson-chamber photographs which would possiblly be interpreted as due to such stable or meta-stable proton-isobars. Anderson and Neddermeyer(13) have namely obtained two such tracks which are hardly accounted for as belonging to any of known elementary particles.

One of them* has a range corresponding to the energy of 1.5 *MeV*, if the particle were a proton. But if this track were really due to a proton, its radius of curvature had to be 20 cm. The measured radius is, however, only one third of 20 cm.

The other thick track* has $H\rho$ of 1.4×10^5 Gauss cm. (H is the applied magnetic field, and ρ the radius of curvature of the track). If one assumes that this track is due to a proton, this $H\rho$ corresponds to the energy of 1 *MeV*. If, now, this assumption were true, the range of the track had to be 2 cm. The observed length of the track was, however, more than 5 cm.

---

(13) C. D. Anderson and S. H. Neddermeyer: P. R., **50** (1936), 263.

* Anderson and Neddermeyer. Fig. 12.

We shall now examine whether these tracks can be interpreted as belonging to our proton-isobar, which has, according to our theory, a mass roughly equal to the proton mass and a charge $-e$ or $2e$, or eventually some other integral multiple of e.

We examine first the second photograph of Anderson and Neddermeyer.* We assume, thus, the charge of the particle to be xe, $x=1, 2, 3, \dots$. Since $H\rho$ is proportional to the momentum of the particle and inversely proportional to the charge, the energy of this particle must be $x^2 \times 1\,MeV$. Next, when the mass is constant, the range of the particle with a specifid energy is inversely proportional to the square of the charge. Therefore, the range of the particle under consideration is given by

$$R=\frac{1}{x^2}R_p(x^2\times 1\,MeV), \qquad (4\cdot 21)$$

where $R_p(x^2\times 1\,MeV)$ is the range of the proton of the energy $x^2\times 1\,MeV$.

If we put $x=2$ in $(4\cdot 21)$, we obtain $R=5.8$ cm, since we have $R_p(4MeV)=23$ cm. This result does not at least contradict with the experimental result: $R>5$ cm. Thus the second track observed by Anderson and Neddermeyer does not, at least, exclude the interpretation that this track is due to the proton-isobar with charge $2e$.

If we apply the same consideration to the particle of the first photograph,* a very large x is required. In table I are given the values of ρ for several values of x calculated by using the observed value of the range. Thus, for $x=1$ the table gives $\rho=20$ cm as mentioned above. The table shows that ρ is still larger than $20/3$ cm even when x is so large as 7. Thus the first track seems to be hardly interpreted as belonging to our proton-isobar.

Table I. Values of the radius of curvature for various values of x, the charge of the particle being xe.

$x=$	1	2	3	4	5	6	7	
$\rho=$	20×1	$\times 0.75$	$\times 0.63$	$\times 0.55$	$\times 0.50$	$\times 0.46$	$\times 0.43$	cm

---

* Ibid. Fig. 12.

No heavy tracks having negative charges have been yet observed. Thus there is no evidence for the existence of the proton-isobar with the charge -1. But such a particle will, even if it does exist, be captured soon by the coulomb attraction into the atomic shell, so that there will be less chance to be observed in a Wilson chamber.

In concluding the article, we notice the following fact. If the coupling between nuclear particle and mesotron field is really so strong that the stable or meta-stable proton-isobars can exist, it is probable that also other atomic nuclei have several state each corresponding to the different state of motion of the mesotrons bound in the nucleus. This situation corresponds to the existence of various electronic states in the molecule. Then these excited "mesotronic states" will probablly have the excitation energies of the order of $10^8 \sim 10^9\,eV.$ over the ground state. The energy supply of this ammount will then cause the mesotronic excitation of a nucleus and, as this state will be unstable, this process will be followed by an emission of slow mesotrons. It is also probable, according to Franck-Condon principle, that the mesotronically excited nucleus has a very high temperature so that an evaporation of protons and nutrons will takes place after the mesotronic excitation. In the case where this excitation is caused by an incident photon, it is probable that the cross-sections for such processes are rather large compared with the cross-sections of the ordinary nuclear photo-effects because the latter cross-sections are exceptionally small owing to the lack of dipole moment in the nuclei. Such phenomena would have some bearing on the production of slow mesotrons as well as slow protons and neutrons in the cosmic rays.

This article was prepared as the report to be read at the symposium on the mesotron theory which was to be held in June 1943. Since this meeting has been prolonged, the article has been published here. In working out the article I am indebted much to Mr. T. Miyazima, Mr. H. Jen, Mr. S. Kanesawa and Mr. T. Tachi for various discussions and assistances. I owe much also to Mr. M. Taketani in preparing this article. I wish to express my sincere thanks to these colleagues of mine.

Department of Physics,
Tokyo Bunrika Daigaku.

Progress of Theoretical Physics Vol. II, No. 2, Mar.~Jun., 1947.

On Dirac's General Transformation Function(1) I.

By Satosi WATANABE.

(Received March 22, 1947)

§ 1. Introduction.

Dirac introduced,(2) some years ago, the idea of "general transformation function" in connection with Lagrange's extremum principle in quantum mechanics, neither giving any rigorous mathematical deduction nor caring much for the condition of existence for such a quantity. Yukawa(3) was the first in this country to imagine a possible important role this quantity would play in the future theory of physics.

The aim of the present paper is to formulate mathematically the so-called general transformation function in both classical and quantum mechanics, and especially to discuss its existence problem. A new point of view will be developed in this paper, according to which physical phenomena are traced in any arbitrary direction in the space-time, contrarily to the ordinary physical theories which deal only with their development in time. Our formalism will show clearly to what extent time can be treated symmetrically with space coordinates in the framework of the current relativistic quantum theory of fields. The final result of the so-called "Uebermehrzeittheorie," which Tomonaga(4) has arrived at rather by a detour, will turn out to be essentially identical with a special expression of our formalism. Some suggestions will be made at the end of the paper as to a possible future revision of the field theory.

Part I will contain mathematical preliminaries and the classical formulation of the general transformation function, and Part II the quantum mechanical formulation and a discussion of the existence problem.

---

(1) Read on Nov. 19th 1944 before the Symposium on the theory of elementary particles at the Second Faculty of Technology, Tokyo Imperial University, Chiba. The essential ideas of this paper were published in Japanese, Kagaku. 14 (1944), 138.

(2) P. A. M. Dirac, Phys. Z. USSR., 3, (1933), 64.

(3) H. Yukawa, Kagaku (in Japanese), 12, (1942), 251, 282, 322.

(4) S. Tomonaga, Progr. Theor. Phys. 1, (1946), 27.
 " , Bull. I. P. C. R. (in Japanese), 22, (1943), 545.

§ 2. Mathematical Preliminaries.

i) Notations concerning functional analysis.

If a quantity y is determined when a function $x(\xi)$ is given throughout a certain continuous domain of the independent variable ξ, y is called a functional of the function $x(\xi)$ and written

$$y=f[x(\xi)]\,. \tag{2·1}$$

y may be regarded as a function of a continuously infinite number of independent variables x which are labelled by the index ξ. If there are a continuously infinite number of such y's, which are labelled by a parameter η, we denote it by

$$y(\eta)=f[x(\xi)\,;\,\eta]\,, \tag{2·2}$$

y being a functional of what stands to the left of the semicolon, and an ordinary function of what stands to the right of the semicolon. It is easy for physicists to understand the meaning of the derivative of the functional $y[x(\xi)]$ with respect to its argument function x, if we write

$$y'[x(\xi)\,;\,\xi_1]=\frac{\vartheta y}{\vartheta x(\xi_1)}=\lim_{\varepsilon\to 0}\frac{f[x(\xi)+\varepsilon\delta(\xi-\xi_1)]-f[x(\xi)]}{\varepsilon}\,, \tag{2·3}$$

where $\delta(\xi-\xi_1)$ is Dirac's δ–function with argument $\xi-\xi_1$. Note well that y in (2·1) is a number, whereas y' in (2·3) is a function of ξ_1, ξ_1 being the point at which the differentiation is made. It is worth-while to see how the derivative is changed when the independent variable ξ is transformed into another $\zeta=\zeta(\xi)$:

$$\frac{\vartheta y}{\vartheta x(\zeta_1)}=\frac{\vartheta y}{\vartheta x(\xi_1)}\left(\frac{d\xi}{d\zeta}\right)_{\zeta=\zeta_1}\,, \tag{2·4}$$

where $\zeta_1=\zeta(\xi_1)$. This rule is easily verified if we remember that $\delta(\zeta)d\zeta=\delta(\xi)d\xi$, which follows from the definition of the δ–function.

Repeating the operation (2·3), we obtain the derivatives of higher orders. For instance, the second derivative is given by

$$y''[x(\xi)\,;\ \xi_1,\ \xi_2]=\frac{\vartheta^2 y}{\vartheta x(\xi_1)\vartheta x(\xi_2)}=\lim_{\varepsilon\to 0}\frac{y'[x(\xi)+\varepsilon\delta(\xi-\xi_2)\,;\ \xi_1]-y'[x(\xi)\,;\ \xi_1]}{\varepsilon}\,, \tag{2.5}$$

in which we can exchange ξ_1 and ξ_2 without altering the result.

The case we often meet with is that in which y is given as a definite integral:

$$y[x(\xi)]=\int\varphi(x,\ x',\ \xi)d\xi\,. \qquad \left(x'=\frac{dx}{d\xi}\right) \tag{2.6}$$

In this case, we obtain the formulae:

$$y'[x(\xi)\,;\ \xi_1]=\int\left\{\frac{\partial\varphi}{\partial x}\delta(\xi-\xi_1)+\frac{\partial\varphi}{\partial x'}\delta'(\xi-\xi_1)\right\}d\xi\,, \tag{2.7}$$

$$\begin{aligned}y''[x(\xi)\,;\ \xi_1,\ \xi_2]=\int\Big\{&\frac{\partial^2\varphi}{\partial x^2}\delta(\xi-\xi_1)\delta(\xi-\xi_2)\\&+\frac{\partial^2\varphi}{\partial x\partial x'}[\delta'(\xi-\xi_1)\delta(\xi-\xi_2)+\delta(\xi-\xi_1)\delta'(\xi-\xi_2)]\\&+\frac{\partial^2\varphi}{\partial x'^2}\delta'(\xi-\xi_1)\delta'(\xi-\xi_2)\Big\}d\xi\,,\end{aligned} \tag{2.8}$$

of which the first one can also be written in a more familiar form:

$$y'[x(\xi)\,;\ \xi_1]=\left|\frac{\partial\varphi}{\partial x}-\frac{d}{d\xi}\frac{\partial\varphi}{\partial x'}\right|_{\xi=\xi_1}. \tag{2.9}$$

We must, however, remember that not all the functionals we deal with in this paper are of the type (2·6).

Returning to the general case, we may extend the Taylor expansion theorem in a form applicable to functionals in the following manner:

$$\begin{aligned}y[x_0(\xi)+\varepsilon x(\xi)]=y[x_0(\xi)]&+\varepsilon\int y'[x_0(\xi)\,;\ \xi_1]x(\xi_1)d\xi_1\\&+\frac{\varepsilon^2}{2}\iint y''[x_0(\xi)\,;\ \xi_1,\ \xi_2]x(\xi_1)x(\xi_2)d\xi_1 d\xi_2\\&+\cdots\cdots,\end{aligned} \tag{2.10}$$

where the integrations are extended over the domain of definition of the independent variable ξ.

We have here considered only functionals containing one argument function $x(\xi)$; the necessary generalization for functionals of more than one argument function will involve nothing essentially new.

ii) Curvilinear coordinates in the Minkowski spece.

We introduce first the rectangular coordinates $x^1=x_1$, $x^2=x_2$, $x^3=x_3$, $x^0=-x_0$, so that the line element ds of the Minkowski space may be written in the form:

$$ds^2=dx^{1^2}+dx^{2^2}+dx^{3^2}-dx^{0^2}=dx_\mu dx^\mu\,, \qquad (2\cdot11)$$

where the current summation convention for the index appearing twice is adopted.

We now take four real functions ξ^μ of x^1, x^2, x^3, x^0 ·

$$\xi^\mu=\xi^\mu(x)\,, \qquad (2\cdot12)$$

where x without index stands for all the four x^μ. We may use ξ^μ as coordinates, unless the Jacobian determinant

$$\frac{\partial(\xi^1,\ \xi^2,\ \xi^3,\ \xi^0)}{\partial(x^1,\ x^2,\ x^3,\ x^0)} \qquad (2\cdot13)$$

vanishes in a continuous domain.

The x^μ–component n^μ of the normal to the surface $\xi^\nu=$const is given by

$$n^\mu \propto \frac{\partial\xi^\nu}{\partial x_\mu}\,. \qquad (2\cdot14)$$

According as the quadratic absolute value:

$$n_\mu n^\mu \qquad (2\cdot15)$$

of n^μ is positive, negative or zero, we say that the surface $\xi^\nu=$const is temporal, spatial or isotropic at the considered point. If the surface $\xi^\nu=$const osculates the light cone, the normal to the surface at the point of osculation has absolute value equal to zero; the normal to the light cone lying always in the light cone itself.

We may speak of orthogonal curvilinear coordinates, if we have everywhere

$$\frac{\partial \xi^{\nu}}{\partial x_{\mu}} \frac{\partial \xi^{\kappa}}{\partial x^{\mu}} = 0 . \qquad \nu \neq \kappa \tag{2·16}$$

We do not, however, always assume the orthogonality of coordinates in the following discussion.

If the (3–dimensional) surface $\xi^{\nu}=\text{const}$ is partly temporal and partly spatial, there will be a (2–dimensional) curve, on which the surface is isotropic, i. e. a curve on which

$$\frac{\partial \xi^{\nu}}{\partial x_{\mu}} \frac{\partial \xi^{\nu}}{\partial x^{\mu}} = 0 \tag{2·17}$$

holds. This relation (2·17) combined with the orthogonality condition (2·16) gives readily

$$\frac{\partial (\xi^1, \xi^2, \xi^3, \xi^0)}{\partial (x^1, x^2, x^3, x^0)} = 0 \tag{2·18}$$

on the above said curve. The geometrical meaning of this result is easy to understand. It is also clear that the light cone cannot be used as one of the four mutually orthogonal coordinates. If one of the four orthogonal coordinates ξ^{μ}, say ξ^0, is everywhere temporal, the other three ξ^i $(i=1, 2, 3)$ will be everywhere spatial.

Example: Let us take coordinates:

$$\begin{aligned} \xi^1 &= x^{1^2} + x^{0^2} , \\ \xi^0 &= x^1 x^0 , \end{aligned} \tag{2·19}$$

which are obviously orthogonal. The Jacobian is given by

$$\frac{\partial (\xi^1, \xi^0)}{\partial (x^1, x^0)} = 2 (x^1 + x^0)(x^1 - x^0) , \tag{2·20}$$

which vanishes on the light cone.

§ 3. Variation Principle.

There may coexist different kinds of fields, either tensorial or spinorial. Let u^{α} be a component of any kind of field, α specifying the component and the kind of the field. $(\alpha=1, 2, \ldots)$. If α is dropped from the symbol, it is only to save writing all the u's.

We do not deal with the transformation character of u with respect to the coordinate changes, which will constitute an independent problem. We denote the derivative of u^a with respect to x by

$$u^a_{x\mu} = \frac{\partial u^a}{\partial x^\mu}. \tag{3·1}$$

We postulate a variation principle for the integral taken over a definite four dimensional volume:

$$I[u(x)] = \int \mathscr{L} dx^1 dx^2 dx^3 dx^0, \tag{3·2}$$

where the integrand $\mathscr{L}$, the Lagrangian, is a certain function of u^a, $u^a_{x\mu}$ ($a=1, 2, \ldots$; $\mu=1, 2, 3, 0$) and x^μ ($\mu=1, 2, 0, 3$):

$$\mathscr{L} = \mathscr{L}(n, u_x, x). \tag{3·3}$$

From the extremum condition:

$$\vartheta I = 0, \tag{3·4}$$

where I is considered as a functional of the functions $u(x)$, follow the wave equations:

$$\frac{\vartheta I}{\vartheta u^a(x)} = \frac{\partial \mathscr{L}}{\partial u^a} - \frac{\partial}{\partial x^\mu} \frac{\partial \mathscr{L}}{\partial u^a_{x\mu}} = 0, \quad (a=1, 2, \ldots) \tag{3·5}$$

if the variation of u's on the boundary of the volume of integration is supposed to vanish.

Using a system of curvilinear coordinates $\xi = \xi(x)$, we introduce a new Lagrangian L by

$$L(u, u_\xi, \xi) = \mathscr{L} \frac{\partial(x)}{\partial(\xi)}, \tag{3·6}$$

where

$$u^a_{\xi\mu} = \frac{\partial u^a}{\partial \xi^\mu}; \quad \frac{\partial(x)}{\partial(\xi)} \equiv \frac{\partial(x^1, x^2, x^3, x^0)}{\partial(\xi^1, \xi^2, \xi^3, \xi^0)}. \tag{3·7}$$

Then the stationary condition (3·4) will give the wave equations:

$$\frac{\vartheta I}{\vartheta u^a(\xi)} = \frac{\partial L}{\partial u^a} - \frac{\partial}{\partial \xi^\mu} \frac{\partial L}{\partial u^a_{\xi\mu}} = 0 \qquad (a=1, 2 \ldots) \tag{3·8}$$

Equations (3·8) are not the covariantly transformed forms of the original equations (3·5) for the new coordinate system, because we do not effect any transformation of the components u. Equations (3·8) are the same equations as (3·5), so long as the Jacobian $\partial(x)/\partial(\xi)$ is finite. This is easily seen if we notice that

$$\frac{\partial}{\partial\xi^\mu}\left\{\frac{\partial\xi^\mu}{\partial x^\nu}\,\frac{\partial(x)}{\partial(\xi)}\right\}=0\,. \tag{3·9}$$

In (3·8) the integral $\boldsymbol{I}$ is considered as a functional of the functions $u(\xi)$, i. e.

$$\boldsymbol{I}[u(\xi)]=\int L(d\xi)^4\,. \tag{3·10}$$

Now we want to pass to an analogy with the formalism of the mechanics of mass-points. For that purpose, we must take out one of the four coordinates ξ, say ξ^0, and assign to it a role similar to that of time in the ordinary mechanics. We shall call this coordinate:

$$s\equiv\xi^0 \tag{3·11}$$

the "observation parameter." We observe physical phenomena not exclusively in their development in time, but in their development in any arbitrary direction in space-time. We shall discuss, at the end of the paper, how far the observation parameter can be arbitrarily chosen without interfering with the current theory of relativity and quantum mechanics

Using a Lagrangian functional $\boldsymbol{L}$ which we obtain by integrating L over three variables ξ^1, ξ^2, ξ^3:

$$\boldsymbol{L}[u,\ u_s;\ s]=\int L(u,\ u_\xi,\ \xi)\,d\xi^1\,d\xi^2\,d\xi^3\,, \tag{3·12}$$

we can write the fundamental principle (3·4) in a form similar to the Hamilton principle of point mechanics:

$$\vartheta\int \boldsymbol{L}\,ds=0\,, \tag{3·13}$$

from which follows

$$\frac{\vartheta\boldsymbol{L}}{\vartheta u^\alpha}-\frac{\partial}{\partial s}\,\frac{\vartheta\boldsymbol{L}}{\vartheta u^\alpha_s}=0\,. \tag{3·14}$$

Since we have obviously

$$\left.\begin{aligned} &\frac{\vartheta \boldsymbol{L}}{\vartheta u^\alpha}=\frac{\partial L}{\partial u^\alpha}-\frac{\partial}{\partial \xi^i}\frac{\partial L}{\partial u^\alpha_{\xi^i}}, \qquad (i=1,2,3) \\ &\frac{\vartheta \boldsymbol{L}}{\vartheta u^\alpha_s}=\frac{\partial L}{\partial u^\alpha_s}, \end{aligned}\right\} \tag{3·15}$$

equation (3·14) is equivalent to equation (3·8).

Equations (3·13) (3·14), having a similar form to the Hamilton and Lagrange principles of point mechanics, allow us to pass to the canonical formalism. Introducing the moment π_α conjugate to u^α by

$$\pi_\alpha=\frac{\partial L}{\partial u^\alpha_s}\left(=\frac{\vartheta \boldsymbol{L}}{\vartheta u^\alpha_s}\right) \tag{3·16}$$

and the Hamiltonian density function by

$$H=\pi_\alpha u^\alpha_s-L, \tag{3·17}$$

we can define the Hamiltonian functional $\boldsymbol{H}$ by

$$\boldsymbol{H}[u,\pi;s]=\int H d\xi^1 d\xi^2 d\xi^3=\int \pi_\alpha u^\alpha_s d\xi^1 d\xi\ \ d\xi^3-\boldsymbol{L}. \tag{3·18}$$

The canonical equations can now be written, following the same deduction as in point mechanics, in the form:

$$\left.\begin{aligned} &\frac{\partial u^\alpha}{\partial s}=\frac{\vartheta \boldsymbol{H}}{\vartheta \pi_\alpha} \\ &\frac{\partial \pi_\alpha}{\partial s}=-\frac{\vartheta \boldsymbol{H}}{\vartheta u_\alpha}. \end{aligned}\right\} \tag{3·19}$$

In the definition (3·16) of π_α, the functional $\boldsymbol{L}$ is differentiated with respect to the argument function u^α_s, which in turn is regarded as a function of the three independent variables ξ. If we use a new set of independent variables ξ':

$$\xi'^i=\xi'^i(\xi^1,\xi^2,\xi^3) \qquad (i=1,2,3) \tag{3·20}$$

instead of ξ, we obtain as the conjugate moments

$$\pi'_\alpha=\frac{\partial L'}{\partial u^\alpha_s}=\frac{\partial L}{\partial u^\alpha_s}\,\frac{\partial(\xi^1\xi^2\xi^3)}{\partial(\xi'^1,\xi'^2,\xi'^3)}. \tag{3·21}$$

[See above (2·4) in the mathematical preliminary.] If we perform a transformation $s=s(s')$, we get

$$\pi'_\alpha=\frac{\partial L'}{\partial u^\alpha_{s'}}=\frac{\partial L}{\partial u^\alpha_s}\,\frac{ds'}{ds}\,\frac{ds}{ds'}=\pi_\alpha \tag{3·22}$$

showing that the conjugate moments remain unchanged, unless the surface form $s=$const is altered. [L' is the transformed Lagrangian function.]

Hence we conclude that the quantity:

$$\pi_\alpha\, d\xi^1\, d\xi^2\, d\xi^3 \tag{3·23}$$

is an invariant for transformations which do not change the surface $s=$const.

The formalism of this section is a generalization, to the curvilinear coordinates, of the formalism which Heisenberg and Pauli used in their quantum theory of fields.⁽¹⁾ The essential difference, however, lies in the fact that we describe here the field phenomena in their development along the observation parameter.

We want to add a remark with regard to the applicability of our formalism. In order that we may pass from the Lagrange formalism, in which u, u_s are the variables, to the canonical formalism, in which u, π are the variables, it must be possible to solve definitely $\pi_\alpha=\pi_\alpha(u,\ u_s)$ (3·16) with respect to u_s, irrespective of the values of ξ. In other words, all the π's should be independent of one another everywhere, which amounts to the condition:

$$\frac{\partial(\pi_1,\ \pi_2,\ \pi_3,\dots)}{\partial(u^1_s,\ u^2_s,\ u^3_s\dots)}\neq 0 \tag{3·24}$$

or, in virtue of (3·16),

$$\det\left(\frac{\partial^2 L}{\partial u^\alpha_s\,\partial u^\beta_s}\right)\neq 0 \tag{3·25}$$

where the determinant has rows and columns labelled by u and β.

It would be a tempting point of view to consider π as more basic physical quantities than u_s and to attribute more significance to the deduced canonical formalism than to the Lagrange formalism which was the starting point. Then the failure of the condition (3·25) would rather mean the

---

(1) **Heisenberg and Pauli, ZS. f. Phys. 56, (1929) 1.**

inadequacy of the Lagrangian formalism.

Example: For the Lagrangian:

$$\mathscr{L}=\frac{\partial u}{\partial x^{\mu}}\frac{\partial u}{\partial x_{\mu}} \tag{3·26}$$

the condition (3·25) becomes

$$\det=\frac{\partial \pi}{\partial u_s}=2\frac{\partial s}{\partial x^{\mu}}\frac{\partial s}{\partial x_{\mu}}\frac{\partial (x)}{\partial (\xi)}\neq 0 \tag{3·27}$$

If the coordinates are chosen so that the Jacobian $\partial(x)/\partial(\xi)$ does not vanish anywhere, this condition may be written

$$\frac{\partial s}{\partial x^{\mu}}\frac{\partial s}{\partial x_{\mu}}\neq 0 \tag{3·28}$$

which means that the normal to the surface $s=\text{const}$ must not be an isotropic vector. Since the light-cone is isotropic everywhere in the space, the condition fails at the points where the surface $s=\text{const}$ osculates the light-cone. If we use, however, an orthogonal coordinate system, the Jacobian in (3·27) will become infinite at such points. See (2·18). For instance, if we use the coordinates (2·19), we have

$$\frac{\partial s}{\partial x^{\mu}}\frac{\partial s}{\partial x_{\mu}}={x^{1}}^{2}-{x^{0}}^{2} \tag{3·29}$$

and

$$\frac{\partial (x)}{\partial (\xi)}=1/2({x^{1}}^{2}-{x^{0}}^{2}), \tag{3·30}$$

so that

$$\det=1\,. \tag{3·31}$$

§ 4. Theory of Canonical Transformation.

The condition that a transformation:

$$\left.\begin{aligned} u^{\alpha}&=u^{\alpha}[U,\ \Pi\ ;\ \xi,\ s]\\ \pi_{\alpha}&=\pi_{\alpha}[U,\ \Pi\ ;\ \xi,\ s]\end{aligned}\right\} \tag{4·1}$$

from the canonical set of variable functions u, π to a new set of variable functions U, Π be a canonical one, is given by

$$\pi_\alpha = \frac{\partial W}{\partial u^\alpha}, \qquad \Pi_\alpha = -\frac{\partial W}{\partial U^\alpha} \tag{4·2}$$

$$K = H + \frac{\partial W}{\partial s}, \tag{4·3}$$

with the help of a generating functional:

$$W[u, U; s], \tag{4·4}$$

where K is the Hamiltonian functional for the new set of canonical variable functions.

The equation, which in our case corresponds to the Hamilton-Jacobi equation, can be obtained by anulling the transformed Hamiltonian functional K, i. e. by putting

$$H\left[u, \frac{\partial W}{\partial u}; s\right] + \frac{\partial W}{\partial s} = 0. \tag{4·5}$$

The functional equations, which we meet ordinarily (for instance integral equations) are relations aiming to find out a function which should satisfy the said equation. Equation (4·5), on the contrary, is a relation requiring to find out a functional W, irrespective of the form of the argument functions u. A perfect solution:

$$W[u, \mu; s] \tag{4·6}$$

of equation (4·5) will then contain a set of constant functions $\mu^\alpha(\xi^1, \xi^2, \xi^3)$. With the aid of such a complete solution, the solution of the wave equation will be given by

$$\pi_\alpha = \frac{\partial W}{\partial u^\alpha},$$

$$\lambda_\alpha = -\frac{\partial W}{\partial \mu^\alpha}, \tag{4·7}$$

where $\lambda_\alpha(\xi^1, \xi^2, \xi^3)$ are integration constant functions.

It is for the time being hopeless to develop a general theory of functional solutions of a functional equation. But in the case of equation (4·5), we can fortunately obtain a solution in analogy to Hamilton's principal function. This solution can be written

$$\boldsymbol{W}[u\{s\},\ u\{s_0\}\ ;\ s]=\int_{s_0}^{s}\boldsymbol{L}\,ds\,. \tag{4·8}$$

$u^\alpha\{s\}$ is a function of ξ^1, ξ^2, ξ^3 in the sense of the definition (1·1); the s in the curved brackets means that the values of u^α of the actual field on the surface $s=\boldsymbol{s}$ should be taken. On the right-hand side, the integral should be taken with respect to the field which is the solution of the wave equation satisfying the boundary conditions given by the values $u\{s\}$, $u\{s_0\}$.

In order to clarify the dependence of $\boldsymbol{W}$ (4·8) on $u\{s\}$, $u\{s_0\}$ and s, we introduce a parameter σ, a function of s, by

$$s=s(\sigma)\,,\quad s_0=s(\sigma_0)\,, \tag{4·9}$$

and write the right-hand side of (4·8) in the form:

$$\int_{\sigma_0}^{\sigma}\boldsymbol{L}\left[u,\ \frac{\partial u}{\partial\sigma}\Big/\frac{ds}{d\sigma}\ ;\ s\right]\frac{ds}{d\sigma}\,d\sigma=\int_{\sigma_0}^{\sigma}\boldsymbol{M}\left[u,\ \frac{\partial u}{\partial\sigma}\ ;\ s,\ \frac{ds}{d\sigma}\right]d\sigma\,. \tag{4·10}$$

If we take the variation of this integral, Euler's expression with regard to u and s will vanish, for the actual solution of the wave equation is substituted in the expression; the remainder will be

$$\vartheta\boldsymbol{W}=\left|\int\frac{\vartheta\boldsymbol{M}}{\vartheta\left(\dfrac{\partial u^\alpha}{\partial\sigma}\right)}\,\vartheta u^\alpha(d\xi)^3\right|_{\sigma_0}^{\sigma}+\left|\frac{\partial\boldsymbol{M}}{\partial\left(\dfrac{ds}{d\sigma}\right)}\,\vartheta s\right|_{\sigma_0}^{\sigma}. \tag{4·11}$$

Since we do not vary the value of s at σ_0, this will become, in consideration of the definition of $\boldsymbol{M}$,

$$\begin{aligned}\vartheta\boldsymbol{W}=&\int\left(\frac{\vartheta\boldsymbol{L}}{\vartheta u_s^\alpha}\right)_{s=s}\vartheta u^\alpha\{s\}(d\xi)^3-\int\left(\frac{\vartheta\boldsymbol{L}}{\vartheta u_s^\alpha}\right)_{s=s_0}\vartheta u^\alpha\{s_0\}(d\xi)^3\\&+\left[\boldsymbol{L}-\int\frac{\vartheta\boldsymbol{L}}{\vartheta u_s^\alpha}\,u_s^\alpha(d\xi)^3\right]\vartheta s\,.\end{aligned} \tag{4·12}$$

Hence

$$\left.\begin{aligned} \frac{\partial W}{\partial u^\alpha\{s\}} &= \left(\frac{\partial L}{\partial u^\alpha_s}\right)_{s=s} = \pi_\alpha\{s\} \\ \frac{\partial W}{\partial u^\alpha\{s_0\}} &= -\left(\frac{\partial L}{\partial u^\alpha_s}\right)_{s=s_0} = -\pi_\alpha\{s_0\} \\ \frac{\partial W}{\partial s} &= L - \int \frac{\partial L}{\partial u^\alpha_s} \cdot u^\alpha_s (d\xi)^3 = -H\,, \end{aligned}\right\} \tag{4.13}$$

which relations show that W of (4.8), in fact, satisfies equations (4.5) (4.7), the values of u and π on the surface $s=s_0$ playing the role of the transformed canonical variables.

The variation in W, as the result of a small change in s, is caused as well by the explicit s in W, as by the intermediary of $u(s)$. So we get, by combining the first and the third relations of (4.13)

$$\frac{dW}{ds} = \frac{\partial W}{\partial s} + \int \frac{\partial W}{\partial u^\alpha} u^\alpha_s (d\xi)^3 = -H + \int \pi_\alpha u^\alpha_s (d\xi)^3 = L\,, \tag{4.14}$$

which is an evident relation in view of the definition of W in (4.8). W is the generating functional which makes possible to deduce the state of the field on $s=s$ from that on $s=s_0$.

The generating functional W, as a definite integral, is identical with the integral (3.2),

$$W = I = \int L\, d\xi^1 d\xi^2 d\xi^3 d\xi^0 = \int \mathcal{L}\, dx^1 dx^2 dx^3 dx^0\,. \tag{4.15}$$

For the generating functional, however, the actual motion i. e. the solution of (3.5) must be substituted in the expression. This quantity is a relativistic invariant, which is determined solely by the values of u on the boundary of the four dimensional volume. W is a functional of the field quantities $u(\xi^1 \xi^2 \xi^3)$ on the boundary which is formed by two hypersurfaces $s=s_0$ and $s=s$.

We can also express this result in a more unified formalism. We choose the observation parameter s in such a way that we may regard the whole boundary surface of the considered volume as a single surface $s=$const, and the integral (4.15) as a functional solely of the field quantities $u\{s\}$:

$$W[u\{s\}\,;\,s]\,. \tag{4.16}$$

The surface s may be either closed or open in a certain direction.

Then the two equations of (4·13) will be united in a single one:

$$\pi_\alpha\{s\} = \frac{\vartheta \boldsymbol{W}}{\vartheta u^\alpha\{s\}}, \tag{4·17}$$

where π_α is still defined by (3·16), but the increasing sense of s will now be directed from inside toward outside of the volume.

The functional $\boldsymbol{W}$, in this interpretation, loses its meaning as a transformation function between two values of the observation parameter; it establishes a relation between u and π at every-point on the boundary. This quantity (4·16) corresponds to Dirac's general transformation function in the pre-quantumtheoretical frame-work.

If we were merely concerned with the final result (4·17), our deduction would be an useless detour. When we deduced the wave equation from the variation principle, we assumed, in fact, that the variation of the u's on the boundary should vanish. If this had not vanish, we should have obtained as the variation of $\boldsymbol{I}$

$$\vartheta \boldsymbol{I} = \int \frac{\partial L}{\partial u^\alpha_s} \vartheta u^\alpha d\xi^1 d\xi^2 d\xi^3 + \int \left(\frac{\partial L}{\partial u^\alpha} - \frac{\partial}{\partial \xi^\mu} \frac{\partial L}{\partial u^\alpha_{\xi\mu}} \right) \vartheta u^\alpha d\xi^1 d\xi^2 d\xi^3 d\xi^0. \tag{4·18}$$

The second integral will disappear for $\boldsymbol{W}$, for the wave equation will then be satisfied. Hence we have

$$\frac{\vartheta \boldsymbol{W}}{\vartheta u^\alpha(\xi^1\xi^2\xi^3)} = \frac{\partial L}{\partial u^\alpha_s} = \pi_\alpha, \tag{4·19}$$

which is equivalent to (4·17). Our deduction in accordance with the theory of canonical transformation will permit us to pass to the quantum theoretical re-formulation of the present theory.

The general transformation function $\boldsymbol{W}$ (4·16) is a functional of $u(\xi^1\xi^2\xi^3)$. If we introduce a transformation of coordinates:

$$\xi'^i = \xi'^i(\xi^1\xi^2\xi^3), \qquad i = 1, 2, 3 \tag{4·20}$$

the functional derivative of $\boldsymbol{W}$ with respect to its argument function u will become, in a similar way to (2·4)

$$\frac{\vartheta \boldsymbol{W}}{\vartheta u^\alpha(\xi')} = \frac{\vartheta \boldsymbol{W}}{\vartheta u^\alpha(\xi)} \frac{\partial(\xi^1\xi^2\xi^3)}{\partial(\xi^{1\prime}\xi^{2\prime}\xi^{3\prime})}, \tag{4·21}$$

showing the same transformation character as the momenta π_α. [See (3·21).] So our final relation (4·17) will obtain an invariant meaning, if we write

$$\pi_\alpha\{s\}\,d\xi^1 d\xi^2 d\xi^3 = \frac{\vartheta W}{\vartheta u^\alpha\{s\}}\,d\xi^1 d\xi^2 d\xi^3\,, \tag{4·22}$$

unless the surface s=constant is changed. (4·22) remains unchanged for a transformation $s'=s'(s)$.

The vital question for the formalism here developed is as to whether it is always possible to satisfy both requirements which the definition of W imposes: namely (i) that the field quantities have the given values on the boundary and (ii) that the u's satisfy the wave equation in the domain. If the Lagrange function and the form of the boundary surface are given in such a way that it is impossible to assign arbitrary values to the u's on the boundary, the functional derivation will lose its sense, and the whole theory will fail. This problem will be discussed at the end of Part II of this paper.

§ 5. Some Properties of the Transformation Functional.

i) Additivity of the general transformation functions.

Analogously to point mechanics, if two transformations

$$u,\ \pi \rightarrow u',\ \pi'$$

and

$$u',\ \pi' \rightarrow u'',\ \pi''$$

are canonical, and if they are derivable from the generating functionals $W'[u, u']$ and $W''[u', u'']$ respectively, the resultant transformation

$$u,\ \pi \rightarrow u'',\ \pi''$$

will also be canonical and derivable from the generating functional

$$W[u, u''] = W'[u, u'] + W''[u', u'']\,. \tag{5·1}$$

The additivity shown by (5·1) is closely connected with that beautiful property of the general transformation functionals, that they are expressed by a volume integral. It is warned, however, against a false conclusion that this additivity should mean that the group of canonical transformations is an Abel group. It goes, of course, without saying, that infinitesimal canonical transformations are commutable.

ii) Infinitesimal transformation.

In order to see how the canonical variables change with an infinitesimal change of the observation parameter, it is more convenient to introduce a generating functional, which is functional of π and U (or u and Π), instead of $\boldsymbol{W}[u, U; s]$, by

$$\boldsymbol{W}'[\pi, U; s]=\boldsymbol{W}-\int u^\alpha\pi_\alpha(d\xi)^3 \tag{5·2}$$

Then the canonical transformation will be given by

$$u^\alpha=-\frac{\vartheta \boldsymbol{W}'}{\vartheta\pi_\alpha}, \qquad \Pi_\alpha=-\frac{\vartheta \boldsymbol{W}'}{\vartheta U^\alpha}. \tag{5·3}$$

If we take $\boldsymbol{L}ds$ as $\boldsymbol{W}$, we get

$$\boldsymbol{W}'=-\int\pi_\alpha U^\alpha(d\xi)^3-\boldsymbol{H}ds, \tag{5·4}$$

and equations (5·3) will then engender the canonical equations of motion:

$$\left.\begin{aligned}\frac{\partial u^\alpha}{\partial s}&=\frac{u^\alpha-U^\alpha}{ds}=\frac{\vartheta \boldsymbol{H}}{\vartheta\pi_\alpha}\\ \frac{\partial\pi_\alpha}{\partial s}&=\frac{\pi_\alpha-\Pi_\alpha}{ds}=-\frac{\vartheta \boldsymbol{H}}{\vartheta u^\alpha}\end{aligned}\right\} \tag{5·5}$$

iii) Conservative system.

If we let the Lagrangian function $\mathscr{L}$ [not L; see (3·6)] contain all the fields under mutual interaction, $\mathscr{L}$ will become, in general, explicitly independent of spatial and temporal coordinates. In such cases, the Lagrangian L will depend on ξ^μ only through the Jacobian $\partial(x)/\partial(\xi)$ in (3·6). If we choose coordinates such that

$$\frac{\partial}{\partial\xi^\nu}\left\{\frac{\partial(x^1x^2x^3x^0)}{\partial(\xi^1\xi^2\xi^3\xi^0)}\right\}=0, \tag{5·6}$$

the Lagrangian L will be independent of the observation parameter.

Let us assume that $\boldsymbol{H}$ does not contain explicitly the observation parameter——a case analogous to the conservative system. If we put

$$\boldsymbol{W}[u; s]=-Es+\boldsymbol{V}[u], \tag{5·7}$$

our Hamilton-Jacobi equation (4·5) will take the form independent of the observation parameter:

$$\boldsymbol{H}\left[u,\ \frac{\vartheta \boldsymbol{V}}{\vartheta u}\right]=E \tag{5·8}$$

A complete solution of this equation $\boldsymbol{V}[\nu,\ \mu]$ will allow us to solve the problem by

$$\left.\begin{aligned} &\pi_\alpha=\frac{\vartheta \boldsymbol{V}}{\vartheta u^\alpha}, \qquad \lambda_\alpha=-\frac{\vartheta \boldsymbol{V}}{\vartheta \mu^\alpha}, \\ &s+s_0=\frac{\vartheta \boldsymbol{V}}{\vartheta E} \end{aligned}\right\} \tag{5·9}$$

E in (5·8) is a constant of motion along the observation parameter. Hence we have, for two values of s,

$$\boldsymbol{H}\{s_0\}=\boldsymbol{H}\{s_1\}. \tag{5·10}$$

If the observation parameter coincides with the time coordinate of any Lorentz frame, the Hamiltonian $\boldsymbol{H}$ will mean the total energy, and the constancy of E is equivalent to the conservation of energy. If, on the contrary, the observation parameter is spatial, the Hamiltonian density H will mean the pressure perpendicular to the surface $s=$const. If, for instance, s coincides with one of the spatial coordinates of a Lorentz frame and $s=x$, $\xi'=y$, $\xi^2=z$, $\xi^3=t$, the Hamiltonian

$$\boldsymbol{H}=\int H\,dy\,dz\,dt \tag{5·11}$$

will then mean the x–component of the total momentum received in course of time by the positive side of the plane $x=$constant from the negative side. The constancy of E along $s=x$ will then be interpreted as expressing the nullity of the x–component of the total momentum, aquired by the volume bordered by two values of x across the both boundaries. When the observation parameter is a curved coordinate, it is difficult to express the meaning of the relation (5·10) in the current terminology of physics.

iv) Perturbation theory.

Let us assume that the Hamiltonian functional $\boldsymbol{H}$ consists of two parts $\boldsymbol{H}_0$ and $\boldsymbol{H}'$:

$$\boldsymbol{H}=\boldsymbol{H}_0+\boldsymbol{H}' \tag{5·12}$$

We can, then, by an adequate transformation, introduce a new set of canonical field quantities, whose dynamical change with the observation parameter is governed only by $\boldsymbol{H}'$. The perturbation theory, which will play an important role in the quantum mechanical formulation, is based on this possibility.

For this purpose, we have only to put the transformed Hamiltonian functional $\boldsymbol{K}$ in (4·3) equal to $\boldsymbol{H}'$, i. e.

$$\boldsymbol{K}=\boldsymbol{H}+\frac{\partial \boldsymbol{W}}{\partial s}=\boldsymbol{H}' \tag{5·13}$$

or, in consideration of (5·12),

$$\boldsymbol{H}_0+\frac{\partial \boldsymbol{W}}{\partial s}=0\,. \tag{5·14}$$

The generating functional $\boldsymbol{W}$ is the solution of the Hamilton-Jacobi equation :

$$\boldsymbol{H}_0\left[u,\ \frac{\vartheta \boldsymbol{W}}{\vartheta u}\ ;\ s\right]+\frac{\partial \boldsymbol{W}}{\partial s}=0 \tag{5·15}$$

The new set of field variables $U^\alpha(\xi)$, $\Pi_\alpha(\xi)$, which are connected with the original ones by

$$\pi_\alpha=\frac{\vartheta \boldsymbol{W}}{\vartheta u^\alpha}\,, \qquad \Pi_\alpha=-\frac{\vartheta \boldsymbol{W}}{\vartheta U^\alpha}\,, \tag{5·16}$$

would be constant functions of ξ^1, ξ^2, ξ^3 if $\boldsymbol{H}'$ were absent. These new variables change with the observation parameter according to

$$\frac{\partial U^\alpha}{\partial s}=\frac{\partial \boldsymbol{H}'}{\partial \Pi_\alpha}\,, \qquad \frac{\partial \Pi_\alpha}{\partial s}=-\frac{\partial \boldsymbol{H}'}{\partial U^\alpha} \tag{5·17}$$

The Hamiltonian $\boldsymbol{H}'$ in (5·12) is supposed to be written in terms of u^α, π_α; the same quantity in (5·17) should be expressed in terms of U^α, Π_α by the help of (5·16).

The Institute of Physical and
Chemical Reserch, Tokyo.

Letters to the Editor

On the Modification of the Hamiltonian of a Nuclear Particle due to its virtual Dissociation.

The wave function of the nuclear particle is assumed to satisfy the Dirac equation in the contemporary theory. If we assume the meson in interaction with the nuclear particle, each state of a nuclear particle must be distorted by a partial superposition of states with coexisting mesons. Thus, the normalized eigenfunction of $H^n+H^m+H^{nm}$ belonging to its eigenvalue E_A is given by

$$\psi_A=\frac{1}{\sqrt{N_A}}\left(u_A-\sum_I u_I\frac{H^{nm}_{IA}}{\varepsilon_{IA}}+\sum_O u_0\sum_I\frac{H^{nm}_{0I}H^{nm}_{IA}}{\varepsilon_{OA}\varepsilon_{IA}}+\sum_{II}u_{II}\sum_I\frac{H^{nm}_{II\,I}H^{nm}_{IA}}{\varepsilon_{IIA}\varepsilon_{IA}}+\cdots\cdots\right)\tag{1}$$

where H^n and H^m are respectively the Hamiltonian of a nuclear particle and mesons, H^{nm} is their interaction. and u_X is a normalized eigenfunction of H^n+H^m belonging to an eigenvalue ε_X. A and O are the states in which there is no mesons, A being not included in the set $\{0\}$. I and II are the states in which there are respectively one meson and two mesons. The denominator ε_{XY} means $\varepsilon_X-\varepsilon_Y$, the matrix element H^{nm}_{XY} stands for $(u_X,\ H^{nm}u_Y)$, and N_A is the norm of the parenthesis of (1). It can be proved, using the explict expression of H^{nm}, that the set $\{0\}$ does not include a state such as $\varepsilon_O=\varepsilon_A$.

If the system is placed in an external field (e.g. a magnetic field), the total Hamiltonian can be written in the form $H=H^n+H^m+H^{nm}+H'$. The matrix element of this Hamiltonian with respect to the set $\{\psi_A\}$ is given by

$$\begin{aligned}(\psi_B,\ H\psi_A)=E_A\delta_{BA}+\frac{1}{\sqrt{N_BN_A}}\Bigg\{&H'_{BA}-\sum_I\left(\frac{H^{nm}_{BI}H'_{IA}}{\varepsilon_{IB}}+\frac{H'_{BI}H^{nm}_{IA}}{\varepsilon_{IA}}\right)\\&+\sum_{I,I'}\frac{H^{nm}_{BI'}H'_{I'I}H^{nm}_{IA}}{\varepsilon_{I'B}\varepsilon_{IA}}\\&+\sum_{I,II}\left(\frac{H^{nm}_{BI}H^{nm}_{I\,II}H'_{IIA}}{\varepsilon_{IB}\varepsilon_{IIB}}+\frac{H'_{BII}H^{nm}_{II\,I}H^{nm}_{IA}}{\varepsilon_{IIA}\varepsilon_{IA}}\right)\\&+\sum_{0,I}\left(\frac{H^{nm}_{BI}H^{nm}_{I0}H'_{0A}}{\varepsilon_{IB}\varepsilon_{OB}}+\frac{H'_{B0}H^{nm}_{0I}H^{nm}_{IA}}{\varepsilon_{OA}\varepsilon_{IA}}\right)+\cdots\cdots\Bigg\}\end{aligned}\tag{2}$$

A linear operator is completely determined by its matrix elements. Accordingly, if we can find an operator U such that $\varepsilon_A\delta_{BA}+(v_B,\ Uv_A)$ is equal to $(\psi_B,\ H\psi_A)$, where v_A is a normalized eigenfunction of H^n belonging to an eigenvalue ε_A, the part of U not included in H' represents an apparent modification of the Hamiltonian of a nuclear particle when mesons are ignored. This method will be referred to as a method of dissociation.

A nuclear particle has its surplus magnetic moment just in this way. Recently I(1) deduced such a surplus Hamiltonian from transitions of the total system. In this method, we consider the time dependence of a wave function $\varphi_A(t)$, of the total system, satisfying an initial condition $\varphi_A(0)=u_A$. It follows from the perturbation theory that

$$(u_B,\ \varphi_A(t))=a_{BA}\frac{e^{-i\varepsilon_Bt}-e^{-i\varepsilon_At}}{\varepsilon_{BA}}+\cdots\cdots,\tag{3}$$

for a state B different from A. The explicit expression of a_{BA} was given by (1·1) in (I).(1) U is calculated by equating a_{BA} and $(v_B,\ Uv_A)$. This method will be referred to as a method of transition. Its applicability is restricted by several conditions. One may doubt the adequacy of the method of transition, for the calculation in (I) involves the perturbation terms up to the third order

The method of dissociation seems to be preferable because of the legitimacy of its principle. Comparing (2) with (1·1) in (I), one find their difference in their denominators. My recalculation shows however that the surplus Hamiltonian of a nuclear particle in a magnetic field is the same in both methods. The term corresponding to the last sum of (2), in which H' is replaced by an interaction between a nuclear particle and a magnetic field, was omitted in (1·3) of (I). The reason for it was not explained in (I). The calculation shows that this term has no contribution to the required magnetic moment.

Another method of calculating the surplus Hamiltonian is an adiabatic method. The theory of the atomic valence is exclusively based on this method. This method is adequate only in the Schrödinger approximation, and it can not be applied to the problem discussed in (1), whereas the nuclear potential can easily be calculated in this way

Department of Industrial Chemistry,
Kyoto Imperial University.
June 21, 1947.
G. Araki

---

(1) G. Araki, Prog. Theor. Phys., 1 (1946), 1. This paper will be referred to as (I)

Some Considerations about the Production of Meson Showers in Matter.

Heitler and his colaborators calculated the cross-section for the production of mesons in high energy nucleon-nucleon collisions.(1) They found thus a very large cross-section, such a large one that it exceeds the square of the range of nuclear forces. Since it is to be expected that two nucleons not approaching nearer than this force range cannot interact at all, Heitler's interpretation for this large cross-section seems not to be adequate. In fact, because their calculation is based on Weizsäcker and Williams' idea, the value they found gives, properly speaking, only the expectation value of the number of produced mesons and not necessarily the probability of a single meson being produced. Thus, the fact that their cross-section exceeds the square of the force range implies that several mesons are produced in a single act.

If we re-interprete Heitler's result in this way, the consideration of Janossy(2) must be modified, since this was made following Heitler's idea. To do this we assume tentatively that the cross-section σ of the production of mesons by a high energy nucleon is always of the order of the square of the force range. The multiplicity ν of the dioduced mesons is then given by dividing Hietler's cross-section by σ. Since σ is larger than the geometrical cross-section per nucleon in the nucleus, the incident nucleon hitting the nucleus produces a meson shower with the probability unity. This results in that the probability of the occurrence of the phenomena in a layer of the thickness x g/cm² is $P(x)=1-\exp(N\sigma_0 x/A^{1/3})$, where A is the atomic weight of the material of the layer, N the number of nucleons contained in its unit volume, and σ_0 the geometrical cross-section per nucleon in the nucleus. This form of $P(x)$ indicates (i) that the meson shower will be observed more frequently under a layer of small atomic weight so far as the thickness x is smaller than the mean free path Λ of the incident nucleon, and (ii) that, when the thickness x becomes larger than Λ, the shower is observed with the same frequency under light as well as heavy materials, provided that the range of the produced mesons are larger than Λ. But if the mean range R of the produced mesons is smaller than Λ, the probability of the meson shower being observed is $1-\exp(N\sigma_0 R/A^{1/3})$

under thick layers. Now the range R will be roughly independent of the material, because the energy of the produced mesons will be independent of the material if we neglect the energy-loss of the mesons which they suffer in the nucleus before they reach its surface. This results in that, as the thickness of the layer is increased, the frequency approaches to a saturation value which is small for heavy materials and large for light ones.

Since, according to the theory of Heitler and others, mesons of energies of the order of several 10^8 eV are expected to be produced most likely, the mean range R will be several 10 g/cm^2. The mean free path Λ, on the other hand, is found to be about $20\,A^{1/3}$ g/cm^2. Thus it is very probable that the above mentioned situation actually occurs.

It was frequently reported that showers under thick layer occur more frequently in lighter materials than in heavier, and this was regarded as a remarkable feature which distinguishes such showers from cascade showers.(3) The consideration given above seems thus to suggest the nature of these non-cascade showers: They are meson showers produced by high energy nucleons.

According to our hypothesis the shower size should be given by the mean number of collisions of the incident nucleon in passing through the nucleus multiplied by the multiplicity. This size is thus $\frac{2}{3}A^{1/3}\nu$. This means that the shower size is proportional to the cube root of the atomic weight. The ratio $38J:3J:2J:1J$ of the shower size found by Messerschmidt(4) under thick layers of Pb, Fe, Al and C seems to be in agreement with this prediction. The data of Messerschmidt enables us further to estimate the multiplicity ν experimentally. From the result $2J$ for Al layer the shower size is estimated to be about 40 in Al and from this we obtain $\nu \approx 10$, which is quite reasonable as compared with $\nu = 3$ obtained from Heitler's result by means of our idea, if we take account of that the value obtained from the experiment will certainly be too large since such a meson shower will be always accompanied by a cascade shower originated from the decay electrons. The transition effect of the penetrating shower observed by several authors shows the existence of such electrons.

If our interpretation of Messerschmidt's experiment is correct, we can estimate the necessary intensity of the high energy nucleons. It is then found that the intensity of such nucleons must amount at sea level to some fraction of percent of the whole hard component there.

(1) J. Hamilton, W. Heitler and H. W. Peng: Phys. Rev. **64** (1943), 78.
W. Heitler and P. Walsh: Rev. Mod. Phys. **17** (1945), 252.
(2) L. Janossy: Phys. Rev. **64** (1943), 345.
(3) H. Euler and W. Heisenberg: Ergebniss der exakten Naturwiss. **17** (1938) 24.
(4) W. Messerschmidt: Zs. f. Phys. **103** (1936), 27.
(5) H. Schindler: Zs. f Phys. **72** (1931), 625.
J. E. Morgan and W. M. Nielsen: Phys. Rev **52** (1937), 564.

Dept. of Physics
Tokyo Bunrika Daigaku
Jan. 10, 1947

S. Tomonaga

ABSTRACTS

of the papers presented to the Symposium on the Theory of Elementary Particles at Physics Department, Kyoto Imperial University, November 21–22–23, 1946 under the auspices of The Physical Society of Japan.

On the Angular Momentum of the Flügge's nuclear liquid drop Model. M. Nogami (*Institute of Physics, Kyushu Imperial University*). Flügge has calculated classically the normal modes of capillary waves on the surface of a perfect liquid drop. Applying his results to the problem of nuclear isomerism, he has obtained conclusions formally very similar to these of Bethe and Weizsäcker. By the quantized liquid drop model, it can be shown, that the angular momentum of the liquid drop, induced by the surface capillary waves has some intimate connections with the number of knodes of these waves, and so the similarity of the Flügge's results to those of Bethe and Weizsäcker is not only formal but of fundamental nature.

On the production of cosmic ray bursts by mesotrons. I. Sato and S. Ozaki (*Department of Physics, Tohoku Imperial University*). As the interaction of vector mesons with the electromagnetic field is stronger than that of scalar mesons, or spin $\frac{1}{2}$ mesons with the electromagnetic field, the field reaction is large in the case of vector mesons. Therefore, when the burst frequency is recalculated consindering the field reaction in the process of Bremsstrahlung of mesons as has been done by Christy and Kusaka, considerable corrections are made with respect to vector mesons. But as the result is larger about ten times than the experimental evidences, it is concluded that vector meson theory is inadequate. On the day of the symposium Mr. Sekido Mr. and Miyazima pointed out to us the papers treated by S. K. Chakrabarty and R. C. Majumdar (Phys. Rev. **65**, (1944)) and Kusaka with respect to this problem.

On the Meson Pair Theory of Nuclear Forces. II. S. Noma (*Institute of Theoretical Physics, Kyoto Imperial University*). The interaction between two nucleons which results from the mutual emission and reabsorption of a pair of spinor mesons is calculated. The pair may consist of any kind of spinor mesons, charged or neutral, but its total charge must not exceed one elementary charge. The positive and the megative mesons are treated symmetrically and the neutral mesons by the Majorana's abreviated theory for the Dirac particle. They are considered as three different states of the same particle. By treating in this way, the interaction with the emission and reabsorption of a pair of the neutral mesons is eliminated for the cases of the vector and tensor type interaction. And only in the case of the pseudo-vector type interaction, we can obtain the potential energy between two nucleons, which is due to the exchange of charge and at the same time gives the right sign of the electric quadrupole moment of the deuteron. The mass difference between the charge and the neutral meson is taken into consideration. The explicit forms of the potential are given for the cases $\gamma=1$ and $\gamma=0$, where γ is the ratio of the masses of the neutral and the charged meson. The difficulty characteristic of the pair theory is avoided in a relativistically invariant way.

Interaction of Elementary Particles. —Mixed Theory of Nuclear Fields.— Y. Tanikawa, M. Hirano and M. Tatsuoka (*Institute of Theoretical Physics, Nagoya Imperial University*). 1. Nuclear Field. We investigated whether the singularity in the static potential of nuclear forces, and the divergence of the nucleon's self-energy could be compatibly removed by means of taking suitable values for the masses and the interaction constants, in a mixed theory of four types of fields. The condition therefore turned out to be that the equation

$$\det\begin{vmatrix} 0 & \delta_b^2-\delta_o^2 & \delta_o^2-\delta_d^2 \\ 3\delta_b^2 & -13(\delta_b^2+\delta_o^2) & 11\delta_o^2-4\delta_d^2 \\ 4-\delta_b^2-2\delta_b, & -(\delta_b^2+\delta_o^2+2\delta_b+2\delta_o+18), & -\delta_o^2-2\delta_o+18 \end{vmatrix}=0$$

hold for values of the sequance of magnitude $\delta_a>\delta_b>\delta_o$, where $\delta_b=M/M_b$, $\delta_o=M/M_o$, $\delta_d=M/M_d$ and M denotes the nucleon mass, while M_b, M_o and M_d are the masses of the vector, pseudo-vector and pseudo-sealar mesons respectively. It was found that above equation does posses roots in the required sequance of magnitude.

2. Meson Decay. In general, when the two types of mesons in mixed field theory differ in mass, the heavier meson can transform into the lighter one with a vidtual pair-creation of nucleons in intermediate states. In Schwinger's modification of the Møller-Rosenfeld theory (where the mass of the vector meson is taken to be larger than that of the pseudoscalar meson), Jauch assumes that the vector meson transforms into the electron and neutrino as usual and obtains 10^{-8} sec. as its lifetime. But, as stated above, the vector meson transforms into the pseudoscalar meson, and consequently its lifetime is much shorter than 10^{-8} sec. Denoting the vector meson by Y, and the pseudoscalar meson by M, the decay processes are

$$Y^+\rightarrow\overline{N}+P\rightarrow\overline{N}+N+M^+\rightarrow M^0+M^+$$

or $$Y^+\rightarrow\overline{N}+P\rightarrow\overline{N}+P'+\nu\rightarrow M^++\nu$$

and for neutral mesons,

$$\begin{tikzcd} & \overline{N}+N \arrow[r] & \overline{N}+P+M^- \arrow[dr] & \\ Y^0 \arrow[ur] \arrow[dr] & & & M+M \\ & \overline{P}+P \arrow[r] & N+M^++\overline{P} \arrow[ur] & \end{tikzcd}$$

$$Y^0\rightarrow\overline{P}+P\rightarrow\overline{P}+P'+\nu\rightarrow M^0+\nu$$

$$\begin{tikzcd} & \nu+\nu'+\nu'' \\ Y \arrow[ur] \arrow[dr] & \\ & M^0+M^{0\prime}+M^{0\prime\prime} \end{tikzcd}$$

computing the lifetime of the vector meson from these processes, taking $M_b=3M_d$, we obtained

$$\tau_0\sim10^{-18}\text{ sec.}$$

On the Exclution Principle of the Assembly of the Systems of Bose-Particles with negative Energy. Y. Tanikawa (*Institute of Theoretical Physics, Nagoya Imperial University*). The problems of Bose-particles with negative energy appear to have become increasing important to the interaction of the elementary particles. For examples, in the theories proposed by Bopp⁽¹⁾ and recently by Dirac, such particles have been introduced in order to overcome the diverging difficulties in the field theory. But by these authors it seems to be overlooked another difficulty arising from negative energy of Bose-particles.

For Fermi-particles, it is wellknown that the difficulty of negative energy could be avoided by the assumption that all the negative energy states are occupied, with one Fermi-particle in each state in accordance with the exclution principle of Pauli, and so a perfect vacuo is a region where all the states of positive energy are unoccupied and all those of negative energy are occupied.

But for Bose-particles such an assumption can not be allowed, because they obey Bose statistics and there does not the exclution principle. Dirac and Pauli(2) has tried in their theories to overcome this difficulty by introducing the idea of the "hypothetical world" where particles have negative energy and negative probabilities and a rule for a transition of results derived originally for the "hypothetical world" into the results for the "actual world" where they have positive energy and positive probabilities. As they say, this rule is however only of the problematical and preliminary character and have no logical foundation in the theory.

In the present paper an alternative method of justifying and extending Dirac's idea is given, based on the following idea. Instead of considering only one system of Bose-particles with negative energy in nature, we assume the existence of the assembly of the identical system of them which obey Fermi statistics as the assembly of the systems. It is needless to say that particles in the system should obey Bose statistics. Such a system may not be formed of definite numbers of particles, we therefore should take the system of the quantized field for it. This is justified from the point of view that the one body problem of the elementary Bose-particle strictly speaking does not exist. This system therefore can include any number of particles with positive and negative energy. It is assumed that a perfect vacuo is a region where all the states of the system in which any number of particles with negative energy, at least one and more, exist are occupied. As the system obeys Fermi statistics, all such states of the system can be occupied with one system in each state in accordance with the exclusion principle just as like as in the case of Fermi particle. One system of Bose particles in our theory corresponds to one Fermi-particle.

The results derived for the systems existing originally in vacuo can be translated in a natural way into the corresponding results for the "actual world." Quite similar is this situation for the physical interpretatjon to that of Dirac's hole theory. We could have interpreted the absorption (emission) of negative Bose particles as the emission (absorption) of positive one. It should be noticable that in our theory the self energy of Bose particles can be reduced to zero.

(1) F. Bopp: Ann. d. Phys. 38 (1940), 345.
(2) W. Pauli: Rev. Mod. Phys 15 (1943), 175.

On the Self-Energy of the Bose Particles. **Y. Tanikawa** (*Institute of Theoretical Physics, Nagoya Imperial University*). The self-energy of the Bose particles, especially photon and meson, arising from the interaction of Fermi particles is investigated. The results are quadratically and logarithmically divergent. As Weisskopf* pointed out in the case of the scalar meson interacting with electromagnetic fields, taking into account the fluctuation energy of vacuo, the quadratically divergent terms can not be reduced to zero.

* V. F. Weisskopf: Phys. Rev. 56 (1939), 73.

On the interaction of elementary particles. II. **T. Miyazima** (*Department of Physics, Tokyo Bunrika Daigaku*). In the previous paper I, it was shown that cut-off factors could be introduced in the generalized Schrödinger equations of the wave fields formulated by Tomonaga and called by him as the "super-many-time theory." The relativistically invariant cut-off theories of Wataghin, Scherzer and Markow were shown to be included in the formalism thus obtained. And it was shown that generalized Schrödinger equations became in general non-integrable corresponding to the finite size of the particles in spacial and time directions. Concerning the cut-off theory, strong-coupling theory has recently been worked out by Pauli and his colaborators, and, on the other hand, the subtraction theory has been extended by Heitler and his colaborators by taking account of the radiation damping. By examining these results, we see that the coupling will not be so strong as the strong-coupling theory holds, nor so weak as the perturbational treatments are permissible. Moreover the cut-off factor will be necessary to avoid divergence difficulties. It will, therefore, be worthwhile to carry out consistent calculation on our generalized Schrödinger equation with the cut-off factors. In doing so, it becomes necessary to fix the method of integration of the equations because the equation is non-integrable. We consider that the method of integration is uniquely fixed by the boundary and initial conditions imposed on the system. On carrying the calculation we must also take into account the radiation reaction and inertia of the fields. We are now treating collision processes of nucleons and mesons under some simplified condition along this line.

On the Elimination of the Auxiliary Condition in the Quantum Electro-dynamics.(1) **S. Hayakawa, Y. Miyamoto and S. Tomonaga** (*Physics Department, Tokyo Bunrika University*). Usual way of eliminating the auxiliary condition in the quantum electrodynamics is performed by separating the field into longitudinal and transverse components.(2) But this distinction of the field is, in general, not relativistic and has the definite meaning only when one specifies a Lorentz frame of reference. The aim of our consideration is to show that the elimination can be performed in a relativistically more satisfactory manner.

For this purpose we first represent the vector potential in a superposition of plane waves whose propagation vectors are denoted by $\boldsymbol{k}$ (thick letters represent four-vectors). Then we introduce three four-vectors $\boldsymbol{l}$, $\boldsymbol{r}$ and $\boldsymbol{s}$ having the properties: $\boldsymbol{l}$ is a null vector whose scalar product with the propagation vector $\boldsymbol{k}$ unity, $\boldsymbol{r}$ and $\boldsymbol{s}$ are unit vectors which are perpendicular to both $\boldsymbol{l}$ and $\boldsymbol{k}$ and with each other. These vectors may depend on $\boldsymbol{k}$ in any way not inconsistent with these conditions. Next we define eight invariants $K=(\boldsymbol{l}\cdot\boldsymbol{A}^{+})$, $L=(\boldsymbol{l}\cdot\boldsymbol{A})$, $R=(\boldsymbol{r}\cdot\boldsymbol{A})$, $S=(\boldsymbol{s}\cdot\boldsymbol{A})$, $K^{+}=-(\boldsymbol{k}\cdot\boldsymbol{A})$, $L^{+}=(\boldsymbol{k}\cdot\boldsymbol{A}^{+})$, $R=(\boldsymbol{r}\cdot\boldsymbol{A})$ and $S^{+}=(\boldsymbol{s}\cdot\boldsymbol{A}^{+})$ using these vectors and the Fourier components $\boldsymbol{A}(\boldsymbol{k})$ and $\boldsymbol{A}^{+}(\boldsymbol{k})$ of the vector potential. Then four pairs of quantities (K, K^{+}), (L, L^{+}), (R, R^{+}) and (S, S^{+}) are proved to be sets of canonically conjugate coordinates and momenta which can be used in order to describe the field.

It is now seen that the part of the field which is described by (R, R^{+}) and (S, S^{+}) is always transverse, while that part which is described by (K, K^{+}) and (L, L^{+}) is not, in general, longitudinal in the ordinary sence. But it is proved that the latter part of the field can be eliminated by the help of the auxiliary condition in the same way as in the usual theory.(3)

The ordinary way of separating the field is contained in ours as the special case corresponding to a spécial choice of $\boldsymbol{l}$.

An advantageous point of our generalized way of separation is that it enables us immediately to find out such separation which corresponds to the ordinary separation carried out in the Lorentz system moving with an arbitrary velocity with respect to the observer. In this case there appear in the Schrödinger equation the retarded Wiechert potentials instead of the Coulombian when one eliminates the generalized longitudinal part of the field.

Our method is conveniently used in treating the problems involving the self-field of a moving electron, because, by taking $\boldsymbol{v}$ equal to the velocity of the electron under consideration, that part of the field which belongs to this electron can be separated immediately from the field of free photons. The calculation of the self-energy of a moving electron, for instance, can be carried out more simply than the usual method.(4) Also the treatment of Bloch and Nordsieck(5) of the radiation of an accelerating electron can be much simplified.

(1) To be published soon in Journ Phys Soc. Japan.
(2) See, for instance, V. Fock: Phys. Zs. USSR. **6** (1934), 425.
(3) V. Fock: loc. cit.
(4) For instance I. Waller: Zs. Phys. **62** (1930), 673.
(5) F. Bloch and A. Nordsieck: Phys Rev. 52 (1937), 54.

Note on the Scattering of a Meson on Colliding with a Nucleon. S. Tomonaga (*Physics Department Tokyo Bunrika University*). The consideration of the previous paper(1) in which the scattering of a meson on colliding with a nucleon was dealt with is generalized so that the higher configurations which more than two mesons are existing are taken into account. In solving the Schrödinger equation in the configuration space we assume that each wave function representing the scattered waves has the singularities of the form $1/\{(K_0^- - \sum K_\nu^- - \sum K_\mu^+) + i\pi\delta(K_0^- - \sum K_\nu^- - \sum K_\mu^+)\}$. In solving this equation the divergence can be avoided if one retains only terms containing a δ-function as factor in the integrand in the integration with respect to the momentum of the meson.* It can be then shown that, when the energy of incident meson is smaller than $N\kappa$, κ being the rest energy of a meson, the configurations in which more than N mesons exist play no role in the problem, so that it suffices only to solve a system of finite number of equations. In order that the conservation of the probability be guaranteed in spite of the omission of the diverging terms, one has to retain only those terms which contain a δ-function as factor also in defining the probability of a specified configuration. This results in that the probabilities of such configurations vanish in which not all existing mesons are free, some of them being bound to the nucleon. Our procedure is so far reasonable that one obtains no scattering at all when one applies this method to the case of neutral scalar meson interacting with a nucleon. This vanishing scattering should be expected because in this case the exact solution of the Schrödinger equation can be obtained by means of a canonical transformation of the field variables, giving rise to no scattering of the incident meson.

(1) S. Tomonaga: Sc. Pap. I.P.C.R., **40** (1942), 73.
* Heitler seems to have proposed a method of avoiding the divergence by omitting some matrix elements of the interaction energy, but his paper is not here obtainable.

On the meaning of the Interaction II. H. Suzuki (*Institute of Theoretical Physics, Kyoto Imperial University*). We consider the interaction of the fields in the form of the constraint equation, not in the Hamiltonian form. This view gives a new light on the difficulties in the present theory. We have seen this method appears in the treatment of the strong-coupling. Now we will consider the meaning of this in connection with the Dirac's λ-limiting process.

Quantum mechanics of circularly polarized photons. G. Araki (*Department of Industrial Chemistry, Kyoto Imperial University*). The unitary transformation of the polarization vectors of photons is studied. The simultaneous eigen-state of the spin angular momentum, the momentum and the energy is determined. It is found that this is the circularly polarized state. The left-circularly polarized photon has the spin of value +1, directed to its momentum, and the value of the spin of the right-circularly polarized photon is −1. The operator creating or annihilating these photons is determined. Further, the eigenfunction, corresponding to the state in which N_l left-circularly and N_r right-circularly polarized photons are present, is calculated. The theory is applied to the special problem of the polarization of the spectral components in the case of the normal Zeeman effect when it is observed along the magnetic field. The complete agreement of the theory with the experiment is obtained.

On the analogy between Born's electrostatic field and compressible flow. I. Imai (*Institute of Physics, Tokyo Imperial University*). Owing to the non-linearity, Born's electromagnetic field is very difficult to deal with. It is remarked here that, in the electrostatic case, Born's field has a perfect formal analogy to the irrotational flow of a non-viscous compressible fluid, so that the various results known in hydrodynamics may be conveniently applied to the Born field. Thus the field strength E and the dielectric constant correspond to the fluid velocity V and the density respectively. While, in the case of compressible flow, the pressure may, in general, be an arbitrary function of the density, Born's electrostatistic field correspond to the special case: An imaginary gas of such a character plays an important part in the theoretical gas dynamics, especially in the two-dimensional flow; it allows of a mathematically exact treatment which is well known as the hodograph method of Karman and Tsien. In this paper, the Karman-Tsien method is translated in the language of electrostatic field in order to deal generally with the two-dimensional Born's electrostatic field. The method is applied to the case of a point charge in a uniform field, and also to the case of two point charges. The exact expressions are obtained for the field strengths, for the force acting on the charges, and for the electrostatic energy of the field. The limiting cases of infinitely great and of vanishingly small distance are then discussed in some detail.

On the energy-momentum tensor. G. Araki (*Department of Industrial Chemistry*). The energy-momentum tensors of the Bose particle of spin 1 are constructed in two different ways, and these two cases are compared. One of these, say case **A**, is the same as the one commonly adopted. The energy, the momentum and the charge of the total system consisting of the Bose particles are the same in both cases, whereas the spin angular momentum and the current (accordingly the magnetic moment) are different in two cases. In case of the vanishing rest mass of the particle, the energy, the momentum and the charge do not include the contribution from the longitudinal components. One case, say case B, does nevertheless not allow the gauge transformation, and this case must be rejected. In case of the non-vanishing rest mass, one can prefer neither of them to the other, as the both cases do not allow the gauge transformation. In the case A, the spin of a particle is given by the Yukawa matrix (vector) in the configuration space, and the spin of a rest particle vanishes in the case B.

On the Hamiltonian form of the general field equations. K. Husimi (*Department of Physics, Osaka Imperial University*). It is a usual procedure to write the classical field equation in the Hamiltonian form in order to translate it into the quantum language, thus failing to incoorporate the relativistic space-time symmetry by preference of a perticular time axis. A form of the Hamiltonian theory of the field, symmetric in space-time coordinates, has already been proposed by H. Weyl, but it appears difficult to interpret it physically. The equations have the divergence form relating to the four conjugate quantities, corresponding to the four axes of space-time, to each component of the field. If integrated over a certain space-time region, it will afford the total flux of the conjugates, taken over the closed perifery of the region. Writing the field equations in this form, integral rather than differential, we get a natual generalization of Maxwellian electrodynamics formalism, and find a certain close analogy with the formalism of general transformation function of Dirac and Yukawa. It will also give an apropriate starting point of a future theory with discrete elements in space-time since it is an integral law.

On the λ-Limiting Process. R. Utiyama (*Department of Physics, Osaka Imperial University*). It is tried to quantize the electromagnetic field in the dispersive medium. If we consider the vacuum as a dispersive medium with a dielectric constant $\varepsilon(k)$ and a magnetic permeability $\mu(k)$, and further we assume $\varepsilon = 1/\mu = 1/\cos(\lambda_0 k_0 - \vec{\lambda}\vec{k})$, then we obtain the so-called λ limiting formalism given by Dirac. Here $\lambda_0 \vec{\lambda}$ is a timelike infinitesimal four vector, $k_0 \vec{k}$ is a wave number four vector of a photon. Furthermore comparing this with the results of the positron theory in respect to the polarization of vacuum we obtain

$$\lambda_0 = \frac{1}{\sqrt{30\pi}} \sqrt{\frac{e^2}{\hbar c}} \frac{\hbar}{mC}.$$

Thus we believe that some physical interpretation of the phenomenological theory is obtained.

On the Non-infinitesimal Commutation Relations. E. Kanai (*Institute of Theoretical Physics, Kyoto Imperial University*). It has been stated that various difficulties appearing in the quantized field theory were due to the infinitesimal commutation-rules first given by Heisenberg and Pauli. Here I take the non-infininitesimal commutation relations which are compatible with the ordinary field equations and are relativistically invariant, then I study the properties of this sort of the quantized field. It is proved for this field to have the particle aspect as in the usual theory, but its energy is not positive definite although is finite. So we must assume the Fermi-statistics in the scalar theory that contradicts the well-known relation between the spin and the statistics.

A Note on the Dirac Space. S. Watanabe. In connection with Pauli's report on Dirac's new method of quantization,(1) some useful notions are introduced to clarify certain confusing points of the method. The anti-self adjoint operator $\boldsymbol{B}$ is defined as having a matrix B which satisfies, in any coordinate system, the relation:

$$\eta B = -B^{+}\eta, \tag{1}$$

η being the metric matrix, and the cross meaning the hermitically conjugate matrix. The eigen-values A' and eigen-states $\psi_{A'}$ of a self-adjoint operator $\boldsymbol{A}$ satisfy

$$(A' - \overline{A''})\eta_{A'A''} = 0, \tag{2}$$

and those of an anti-self-adjoint operator $\boldsymbol{B}$ satisfy

$$(B' + \overline{B''})\eta_{A'A''} = 0. \tag{3}$$

The operator $\boldsymbol{x}$ "associated" to an operator $\boldsymbol{q}$ is defined by the relation: $\boldsymbol{q} = i\boldsymbol{x}$. If $\boldsymbol{q}$ is self-adjoint (anti-self-adjoint), then $\boldsymbol{x}$ is anti-self-adjoint (self-adjoint). When a self-adjoint $\boldsymbol{q}$ has imaginary eigen-values, it is conven ent to use the representation by the real eigen-values of its associated operator $\boldsymbol{x}$. In the case of continuous eigen-values, this representation will give

$$q_{x'x''} = ix'\delta(x' - x''), \tag{4}$$

and from (3)

$$\eta_{x'x''} - \delta(x' + x''). \tag{5}$$

These two relations are identical with equations (26) and (23) of Pauli's report. In that report the notation q is used both in the sense of $\boldsymbol{q}$ and in the sense of x' of our analysis.

---

(1) W. Pauli, Rev. Mod. Phys. 15, 1943, 175.

An Attempt at the Theory of Elementary Particles. H. Yukawa (*Physics Department, Kyoto Imperial University*). As an attempt to get rid of the present difficulties of the theory of elementary particles, we consider material particles such as electrons, protons and neutrons as something different from the electromagnetic and nuclear fields. We start namely from the one body problem for the material particle, in which the field variables are treated as quantities operating upon the unquantized wave function for the particle. In other words, the field quantities become certain functions not only of space and time coordinates, but also of energy-momentum and spin variables for the particle. In particular, they can be expressed as linear functions of the ordinary and isotopic spins, so that there can exist, in addition to the electromagnetic field, only the fields which are accopanied by various types of neutral and charged mesons with spin 0 and 1. Thus an extended mixed field theory is obtained as the immediate consequence of our assumption. Moreover, as we are now free from the restriction that the field should be functions of the space and time coordinates alone, we find ample grounds for introducing the fundamental quantity such as the radius of the elementary particle in a rational way. A simple way of extending the representation of the wave equations in the ordinary 4 dimensional space to that in 8 dimensional phase space is suggested. The next step to the many body problem through the procedure of second quantization of particle wave functions as well as of field quantities is in course of investigation.

Correlation between Mutual Interactions of Elementary Particles. S. Sakata (*Institute of Theoretical Physics, Nagoya Imperial University*). Hitherto, the mutual interactions of elementary particles have been studied separately, with no consideration whatever of the correlation between two or more such interactions. It seems to us that this is precisely one of the chief causes of the difficulties arising in quantum field theory. The fact that Møller and Rosenfeld lately succeeded in removing singularities of nuclear forces by considering the correlation between two meson fields justifies this view-point. Bopp's linear electronic theory is noteworthy in that it gives a finite value to the static selfenergy, and his theory may be interpreted as a mixture of the electromagnetic and neutral vector meson fields. The drawback of this theory is that the dynamical part of the selfenergy diverges, and that the energy of the meson becomes negative. We have introduced, as the field of electron-cohesion, a neutral scalar meson field, and have proved, positron-theoretically, that the total selfenergy remains finite, to the approximation of e^2. By extending this line of thought to protons, a theoretical explanation of the mass difference between proton and neutron can be obtained.

Some Consideration about the Production of Meson Showers in Matter. S. Tomonaga (*Physics Department, Tokyo Bunrika University*). Heitler and his colaborators(1) calculated the cross-section for the production of a meson by a high energy nucleon on coliding with another nucleon. They found thus a very large value for this cross-section, such a large value that it exceeds the square of the range of the nuclear forces. Since it is to be expected that two nucleons not approaching nearer than this force range cannot interact at all, Heitler's interpretation for this large value of the cross-section seems not to be adequate. In fact, because their calculation is based on Weizsäcker and William's idea, the value they obtained gives, properly speaking, only the expectation value of the number of the produced mesons and not necessary the probability of a single meson being produced in the collision. Thus, the fact that the cross-section exceeds the square of the force range imples that several mesons are produced in a single act.

This re-interpretation of Heitler's result makes it necessary to modify the consideration of Janossy(2) who has dealt with the production of meson showers in matter. We thus treat this problem assuming tentatively that the cross-section σ of this phenomena is always of the order of the square of the force range. Then the multiplicity of the produced mesons is given by dividing Heitler's cross-section by σ.

In this way we can discuss the material dependence of the frequency and size of the meson showers produced in thick layers of various materials. The results seems to have bearing on the experimental evidences concering the burst of not too large size observed under thick layers.(3)

(1) J. Hamilton, W. Heitler and H. W. Peng: Phys. Rev. **64** (1943), 78.
W. Heitler and P. Walsh: Rev. Mod. Phys., **17** (1945), 252.
* According to the newly arrived, Bulletin of American Physical Society Bethe seems to be of the same opinion.
(2) L. Janossy: Phys. Rev. **64** (1943), 345.
(3) H. Messerschmidt: Zs. Phys. **72** (1931), 625.

Super-Many-Time Formulation of the Quantum Theory of Wave Fields. S. Tomonaga (*Physics Department, Tokyo Bunrika University*). According to the program proposed some years ago(1) a perfectly relativistic formalism (the so-called super-many-time formalism) is given for the quantum theory of the electromagnetic field interacting with either electron or meson field. In the case of the electron field(2) no essential modification of the previous consideration is necessary excepting that some special considerations are needed owing to the so-called auxiliary condition. It is shown that the elimination of this condition is possible in the relativistically invariant way on the line given in another place.(3) Also the invariance of the theory with respect to the transformation of the gauge can be proved.

In the case of the meson field,(4) on the other hand, a generalization of the previous consideration is necessary because it can not be directly applied to this case by the following two reasons: (i) the interaction-energy density $H_{12}(P)$ is not a scalar, and (ii) the condition $[H_{12}(P), H_{12}(P')]=0$ which is necessary for the integrability of the differential equation $\left\{H_{12}(P)+\frac{\hbar}{i}\frac{\delta}{\delta C_p}\right\}\Psi[C]=0$ does not hold in this case.

These two difficulties can be got over if one uses not $H_{12}(P)$ itself in the generalized Schrōdinger equation but the quantity obtained by supplimenting it by a suitablly chosen term $A_p[C]$ which vanishes when the variable surface C reduces into a plane parallel to the xyz-plane. It can be shown that it is possible to determine this term in such a way that it suppliments the non-scalar term in $H_{12}(P)$ making up a scalar density $H_p[C]=H_{12}(P)+A_p[C]$, and, at the same time, $H_p[C]$ thus obtained satisfies the condition $\left[H_p[C]+\frac{\hbar}{i}\frac{\delta}{\delta C_p}, H_{p'}[C]+\frac{\hbar}{i}\frac{\delta}{\delta C_{p'}}\right]=0$. Then the equation $\left\{H_p[C]+\frac{\hbar}{i}\frac{\delta}{\delta C_p}\right\}\Psi[C]=0$ has the properties: it is integrable, relativistic invariant, and reduces into the ordinary Schrödinger equation when the variable surface C reduces into a plane parallel to the xyz-plane. Thus this equation can be regarded as the generalized Schrōdinger equation required in our theory.

The same method can be applied to the case of the meson field interacting with the nucleon field.(5)

(1) S. Tomonaga: Bull. I.P.C.R. (Riken-Iho) 22 (1943), 545; the English translation: Progr. Theor. Phys., **1** (1946), 27.
(2) Z. Koba, T. Tachi and S. Tomonaga: Progr. Theor. Phys. in press.
(3) S. Hayakawa, Y. Miyamoto and S. Tomonaga: Reported at this meeting. This work will soon appear in Journ. Phys. Soc. Japan.
(4) S. Kanesawa and S. Tomonaga: Progr. Theor. Phys. in press.
(5) Y. Miyamoto: to be published soon.

Progress of Theoretical Physics Vol. II, No. 3, Jul.~Oct. 1947.

On a Reletivistically Invariant Formulation of the Quantum Theory of Wave Fields. II.

— Case of Interacting Electromagnetic and Electron Fields —

Zirô KOBA, Takao TATI and Sin-itiro TOMONAGA.

(Received Nov. 6, 1946)

§ 1. Introduction

In a previous paper with the same title⁽¹⁾ one of the authors has proposed a new formalism of the quantum theory of wave fields which reveals explicitly its relativistic invariance. For this purpose the author has generalized the Schrödinger equation of the system into the following form:

$$\left\{H_{12}(P)+\frac{\hbar}{i}\frac{\delta}{\delta C_P}\right\}\Psi[C]=0. \qquad (1\cdot1)$$

The generalized ψ-vector $\Psi[C]$ in $(1\cdot1)$ is a functional of the independent variable hyper-surface C in the four-dimensional space-time world, and $H_{12}(P)$ is the density of the interaction energy between the fields 1 and 2 at the world point P lying on the three-dimensional surface C. The operation $\delta/\delta C_P$ is the functional partial differentiation of $\Psi[C]$ at the point P, defined by

$$\frac{\delta\Psi[C]}{\delta C_P}=\lim_{\substack{C'\to C\\ \Delta V\to 0}}\frac{\Psi[C']-\Psi[C]}{\Delta V}, \qquad (1\cdot2)$$

where C' is a surface overlapping C everywhere except a small region surrounding the point P, and ΔV is the volume of the small world region enclosed by C and C'.

In our fundamental equation $(1\cdot1)$ the generalized ψ-vector $\Psi[C]$ and its functional derivative are both relativistically invariant concepts. Moreover H_{12} is assumed to be a scalar function of the field quantities. (The case where H_{12} is not a scalar will be discussed in a later paper.*)

---

1) S. Tomonaga: Riken Iho, **22** (1943), 525. English translation. Progr. Theor. Phys., **1** (1946), 40. This paper will be cited as I.

* S. Kanesawa and S. Tomonaga, Progr. Theor. Phys. in press.

These field quantities satisfy, in turn, the field equations for the free fields, which have covariant forms under Lorentz transformations. The commutation relations between these field quantities are expressed in terms of the so-called four-dimensional delta-functions and their derivatives, and are free from any unsatisfactory feature of the ordinary commutation relations. In this way the equation $(1\cdot1)$ has a quite satisfactory four-dimensional space-time form possessed of a definite meaning without referring to any special Lorentz frame.

The equation $(1\cdot1)$ is also integrable when H_{12}'s at any two world points at a finite as well as at an infinitesimal distance apart, one lying outside the light cone of the other are commutable with each other.

The aim of the present paper is to apply our method of formulation to the quantum electrodynamics which deals with the electromagnetic field interacting with the electron field.

Although we have in the case of quantum electrodynamics a perfectly relativistic formalism, that is, Dirac's many-time theory,(2) it will be nevertheless of some importance to formulate the theory also in our formalism, since the former, in which the states of electrons are described in the configuration space, applies only to the case where the number of the electrons does not change, and is thus incapable of dealing with such phenomena as the emission of β-rays or the decay of mesons emitting electrons, without introducing a rather unfavourable complication into the theory at the sacrifice of its beautifulness. It is, therefore, desirable to treat the electrons as a quantized field and not as particles, since the former treatment fits the cases better in which the number of the electrons really changes.

It is already nearly known in I how to apply our formalism to the quantum electrodynamics except the well-known complication due to the existence of the so-called auxiliary condition. One of the main tasks in the present paper is thus to find the auxiliary condition in our formalism and to eliminate it in a relativistically invariant way on the lines which has been reported by Hayakawa, Miyamoto and one of the authors in another place.(3)

---

2) P. A. M. Dirac, V. Fock and B. Podolsky: Phys. Zs. USSR., **2** (1932), 468.

3) S. Hayakawa, Y. Miyamoto and S. Tomonaga: J. Phys. Soc. Japan in press. This paper will be cited as A.

§ 2. Commutation Relations between Field Quantities, Field Equations for Free Fields and the Generalized Schrödinger Equation.

When it is not specially remarked, the same notations are used throughout this paper as in I and A. Furthur, we use the unit system in which the light velocity is unity.

The Lagrange density of the free electromagnetic field is then given by

$$L_{\text{I}} = -\frac{1}{4}(\text{Curl } \boldsymbol{A})^2 - \frac{1}{2}(\text{Div } \boldsymbol{A})^2, \tag{2·1}$$

and that for the free electron field is

$$L_{\text{II}} = i\hbar\psi^*\{(\alpha, \text{ Grad}) + i\varkappa\beta\}\psi, \tag{2·2}$$

where α denotes the four-vector $(1, \vec{\alpha})$, $\vec{\alpha}$ and β being the Dirac matrices. The constant $\varkappa$ in (2·2) is the reciprocal of the Compton wave length of the electron.

In the presence of the interaction between electron and electromagnetic field one has to replace Grad in (2·2) by Grad $-\frac{i\varepsilon}{\hbar}\boldsymbol{A}$, so that for the Lagrange density of the total system we have

$$L = -\frac{1}{4}(\text{Curl } \boldsymbol{A})^2 - \frac{1}{2}(\text{Div } \boldsymbol{A})^2 + i\hbar\psi^*\{(\alpha, \text{ Grad}) + i\varkappa\beta\}\psi \\ + \varepsilon(\psi^*\alpha\psi, \boldsymbol{A}). \tag{2·3}$$

Through the usual procedure we obtain from (2·3) the Hamiltonian density of the system

$$\begin{cases} H = H_{\text{I}} + H_{\text{II}} + H_{\text{I,II}} \\ H_{\text{I}} = \frac{1}{2}\left\{\left(\frac{\partial \vec{A}}{\partial t}\right)^2 - (\text{grad } A_0)^2 + (\text{div } \vec{A})^2 - \left(\frac{\partial A_0}{\partial t}\right)^2 + (\text{curl } A)^2\right\} \\ H_{\text{II}} = -i\hbar\psi^*\{\alpha, \text{ Grad}) + i\varkappa\beta\}\psi \\ H_{\text{I,II}} = -\varepsilon(\psi^*\alpha\psi, \boldsymbol{A}) \end{cases} \tag{2·4}$$

in which H_{I} is the energy density for the electromagnetic field, H_{II} that for the electron field, and $H_{\text{I,II}}$ the energy density of the interaction of these two fields. Having thus obtained the Hamiltonian, we can write down the Schrödinger equation for our system:

$$\left(\bar{H}+\frac{\hbar}{i}\frac{\partial}{\partial t}\right)\Psi=0.$$

According to the general scheme given in I, we introduce the unitary operator U defined by

$$U=\exp\left\{\frac{i}{\hbar}(\bar{H}_{\mathrm{I}}+\bar{H}_{\mathrm{II}})t\right\}$$

and transform the field quantities A, ψ and ψ^* as follows:

$$\begin{aligned}&\boldsymbol{A}\rightarrow U\boldsymbol{A}U^{-1}\\&\psi\rightarrow U\psi U^{-1}\\&\psi^*\rightarrow U\psi^*U^{-1}.\end{aligned}$$

We further transform the Schrödinger functional by means of U:

$$\Psi\rightarrow U\,\Psi.$$

In the following the letters $\boldsymbol{A}$, ψ, ψ^* and Ψ are used to denote these transformed quantities.

These transformed quantites depend on t, while the original ones did not, since there we were dealing with the Schrödinger form of the quantum theory. It can be shown that the transformed field quantities depend on t as if each field were free. Thus they satisfy the field equations for the free fields:

$$\begin{cases}\Box\boldsymbol{A}=0\\\{(\alpha,\ \mathrm{Grad})+i\varkappa\beta\}\psi=0\\\psi^*\{(\alpha,\ \mathrm{Grad})-i\varkappa\beta\}=0.\end{cases}\tag{I}$$

The time dependence of the transformed Ψ-vector, on the other hand, is determined by

$$\left(\bar{H}_{\mathrm{I,II}}+\frac{\hbar}{i}\frac{\partial}{\partial t}\right)\Psi=0\tag{2·5}$$

in which $\bar{H}_{\mathrm{I}}$ and $\bar{\bar{H}}_{\mathrm{II}}$ appear no longer, while the field quantities contained in $\bar{H}_{\mathrm{I,II}}$ have to be considered as satisfying (I).

The field equations (I) together with the ordinary commutation relations, which give the commutator between field quantities at the same instant of time, enable us to calculate the four-dimensional commutation relations, which give the commutators between field quantities at any two world points X and X':

$$\begin{cases} [A^{\mu}(X),\ A^{\nu}(X')]=\dfrac{\hbar}{i}g^{\mu\nu}D_{\mathrm{I}}(X-X') \\ [\psi_{\sigma}(X),\ \psi_{\rho}^{*}(X')]_{+}=W_{\sigma\rho}(X-X') \\ \text{with}\quad W_{\sigma\rho}(X-X')=\left\{\delta_{\sigma\rho}\dfrac{\partial}{\partial t_X}-(\vec{\alpha}_{\sigma\rho},\ \mathrm{grad}_X)-i\varkappa\beta_{\sigma\rho}\right\}D_{\mathrm{II}}(X-X') \\ [\psi_{\rho}(X),\ \psi_{\sigma}(X')]_{+}=[\psi_{\rho}^{*}(X),\ \psi_{\sigma}^{*}(X')]_{+}=0 \\ [A^{\mu}(X),\ \psi_{\rho}(X')]=[A^{\mu}(X),\ \psi_{\rho}^{*}(X')]=0. \end{cases} \tag{II}$$

In (II) D_{I} and D_{II} are the four-dimensional delta-functions belonging respectively to the electromagnetic field and the electron field. They are both invariant with respect to Lorentz transformations. When X and X' are two world points, one of which lies outside the light cone of the other, they have the properties:

$$\begin{cases} D(X-X')=0 \\ \{\partial/\partial x_i\, D(X-X')\}_{t=t'}=0 \qquad \text{for}\quad i=1,\ 2,\ 3 \\ \{\partial/\partial x_0\, D(X-X')\}_{t=t'}=\delta(\vec{x}-\vec{x}') \\ \{\partial^2/\partial x_0^2\, D(X-X')\}_{t=t'}=0 \end{cases} \tag{2・6}$$

From these properties it follows that, when X and X' refer to the same instant of time, the commutation relations (II) reduce to the ordinary ones:

$$\begin{cases} [A^{\mu}(X),\ A^{\nu}(X')]=0, \qquad \left[A^{\mu}(X),\ \dfrac{\partial A^{\nu}(X')}{\partial x_0'}\right]=i\hbar g^{\mu\nu}\delta(\vec{x}-\vec{x}') \\ [\psi_{\sigma}(X),\ \psi_{\rho}^{*}(X')]_{+}=W_{\sigma\rho}(X-X') \\ \quad\text{with}\quad W_{\sigma\rho}(X-X')=\delta_{\sigma\rho}\,\delta(\vec{x}-\vec{x}'). \end{cases} \tag{II$'$}$$

We now generalize the Schrödinger equation (2・5) according to the general scheme developed in I. This gives

$$\left(H_{\mathrm{I,II}}(P)+\frac{\hbar}{i}\frac{\delta}{\delta C_P}\right)\Psi(C)=0. \tag{III}$$

The meanings of $H_{\mathrm{I,II}}(P)$, $\delta/\delta C_P$ and $\Psi[C]$ have been already given in § 1. The interaction energy density $H_{\mathrm{I,II}}(X)$ is here of the form*

$$H_{\mathrm{I,II}}(X)=-\varepsilon(\boldsymbol{J}(X),\ \boldsymbol{A}(X)) \qquad (2\cdot7)$$

with

$$\boldsymbol{J}(X)=\psi^*(X)\alpha\psi(X) \qquad (3\cdot8)$$

which has the physical meaning: four-current density at the point X. Since $\boldsymbol{J}(X)$ is a four-vector, $H_{\mathrm{I,II}}$ of $(2\cdot7)$ is evidenthy an invariant, so that our fundamental equation (III) is relativitically invariant. Further, as we shall prove in the following, $H_{\mathrm{I,II}}(X)$ has the property:

$$[H_{\mathrm{I,II}}(X),\ H_{\mathrm{I,II}}(X')]=0 \qquad (2\cdot9)$$

for any two world points X and X', one lying outside the other's light cone. These points may lie a finite distance apart or may be two adjacent points. The property $(2\cdot9)$ guarantees that our equation (III) is integrable when C is space-like.

We shall now show that the integrability condition $(2\cdot9)$ is really satisfied.

Since $H_{\mathrm{I,II}}$ is an invariant and the commutation relation is also invariant, the left-hand side of $(2\cdot9)$ can be most conveniently calculated in a reference system in which X and X' refer to the same instant of time. Let the components of various quantities in such a coordinate system be denoted by barred suffixes. Then, according to (II′) we have $[A_{\bar{i}}(X),\ A_{\bar{k}}(X')]=0$. so that we obtain first

$$[H_{\mathrm{I,II}}(X),\ H_{\mathrm{I,II}}(X')]=\varepsilon^2[J^{\bar{i}}(X),\ J^{\bar{k}}(X')]A_{\bar{i}}(X)A_{\bar{k}}(X') \qquad (2\cdot10)$$

In order to calculate $[J^{\bar{i}}(X),\ J^{\bar{k}}(X')]$ we use the following theorem which can be proved by a straightforward calculation:

Theorem I. Let U and V be any two fou-row-four-column matrices. Then we have

$$\begin{aligned}[\psi^*(X)U\psi(X),\ \psi^*(X')V\psi(X')]&=\psi^*(X)UW(X-X')V\psi(X')\\&\quad-\psi^*(X')VW(X-X')U\psi(X). \qquad (2\cdot11)\end{aligned}$$

Now, to calculate $[J^{\bar{i}}(X), J^{\bar{k}}(X')]$, we have only to put $U=V=\alpha$ in this formula. Then we obtain, noting $W_{\bar{\rho}\bar{\sigma}}=\delta_{\bar{\rho}\bar{\sigma}}\delta(\vec{x}-\vec{x}')$,

---

* We use the letters X, X', X'', ... to denote arbitrary world points whereas the letters P, P', P'', ... are used to denote world points on C.

$$[J^{\bar{i}}(X),\ J^{\bar{k}}(X')]=\psi^*(X)(\alpha^{\bar{i}}\alpha^{\bar{k}}-\alpha^{\bar{k}}\alpha^{\bar{i}})\psi(X)\delta(\vec{x}-\vec{x}').$$

Substituting this commutator into (2·10), we get

$$[H_{\mathrm{I,II}}(X),\ H_{\mathrm{I,II}}(X')]=\psi^*(X)(\alpha^{\bar{i}}\alpha^{\bar{k}}-\alpha^{\bar{k}}\alpha^{\bar{i}})\psi(X)A_{\bar{i}}(X)A_{\bar{k}}(X)\delta(\vec{x}-\vec{x}')=0,$$

what is to be proved.

§ 3. Auxiliary Condition for the Electromagnetic Potential.

It is well known that our Lagrange function (2·1) gives rise to $\Box\boldsymbol{A}=0$ but does not yield the Maxwell equation. In order that the canonical equations derived from (2·1) have the Maxwell form, one must impose the auxiliary condition

$$\{\mathrm{Div}\ \boldsymbol{A}(X)\}\Psi=0 \qquad (3\cdot1)$$

on Ψ, and admit only such Ψ to represent actual states. When the field is interacting with electrons, however, the condition (3·1) is not compatible with our fundamental equation (III), so we must replace (3·1) by

$$\Xi_X[C]\Psi[C]=0, \qquad (3\cdot2)$$

where $\Xi_X[C]$ has the form

$$\Xi_X[C]=\mathrm{Div}\ \boldsymbol{A}(X)+f_X[C]. \qquad (3\cdot3)$$

The added term $f_X[C]$ is a function of the world point X on one hand and a functional of the variable surface C on the other hand. It must be so determined that the condition (3·2) is compatible with the equation (III). It is thus required that $f_X[C]$ satisfies

$$\left[H_{\mathrm{I,II}}(P)+\frac{\hbar}{i}\frac{\partial}{\partial C_P},\ \mathrm{Div}\ \boldsymbol{A}(X)+f_X[C]\right]=0 \qquad (3\cdot4)$$

for all points X, P being any point lying on C.

Now, since we have

$$[H_{\mathrm{I,II}}(P),\ \mathrm{Div}\ \boldsymbol{A}(X)]=\frac{\hbar}{i}\varepsilon(\boldsymbol{J}(P),\ \mathrm{Grad}_P D_{\mathrm{I}}(P-X)),$$

$$\left[\frac{\hbar}{i}\frac{\partial}{\partial C_P},\ \mathrm{Div}\ \boldsymbol{A}(X)\right]=0,$$

our problem of finding the solution of (3·4) is solved when we can find $f_X[C]$ such that

$$\left\{\begin{array}{l} [H_{\mathrm{I,II}}(P),\ f_X[C]]=0 \\ \left[\dfrac{\hbar}{i}\,\dfrac{\delta}{\delta C_P},\ f_X[C]\right]=-\dfrac{\hbar\varepsilon}{i}(\boldsymbol{J}(P),\ \mathrm{Grad}\ _P D_1(P-X)). \end{array}\right. \qquad (3\cdot5)$$

The second relation of (3·5) gives rise to the differential equation

$$\frac{\delta f_X[C]}{\delta C_P}=-\varepsilon\ \mathrm{Div}_P(\boldsymbol{J}(P)D_1(P-X)) \qquad (3\cdot6)$$

on account of Div $\boldsymbol{J}=0$.

The differential equation (3·6) can be solved immediately by using the following theorem:

Theorem II. Let $\boldsymbol{G}(X)$ be an arbitrary four-vector function of the world point X, and let C be any space-like surface in the space-time world. Further, let $\boldsymbol{N}(P)$ denote the unit vector which is normal to C at the point P on C. The direction of $\boldsymbol{N}$ is such that its contravariant time component is positive (this means that $\boldsymbol{N}$ is pointing to the future). Then we have

$$\frac{\delta}{\delta C_P}\int_C(\boldsymbol{G}(P'),\ \boldsymbol{N}(P'))dF_{P'}=-\mathrm{Div}\ \boldsymbol{G}(P) \qquad (3\cdot7)$$

where the integral on the left-hand side is the surface integral taken over the surface C, $dF_{P'}$ being its surface element at P'.

Proof. According to Gauss' theorem we have, in general,

$$-\int_{\mathrm{Vol}}\mathrm{Div}\ \boldsymbol{G}(X)dV=\int_{\mathrm{Surf.}}(\boldsymbol{G}(P),\ \boldsymbol{N}(P))dF_P \qquad (3\cdot8)$$

where the integral on the left-hand side is the volume integral taken over a four-dimensional spece-time region, and the integral on the right-hand side is the surface integral taken over the surface enclosing this region, $\boldsymbol{N}(P)$ being the outward normal to this surface at the point P. We cosider the space-time region to be that part of the world which lies on the past side of the variable surface C. Then the surface integral on the right-hand side of (3·8) consists of two parts: that taken over the surface C and that over the surface at infinity:

$$-\int_C \mathrm{Div}\ \boldsymbol{G}(X)dv = \int_C (\boldsymbol{G}(P),\ \boldsymbol{N}(P))dF_P + \int_\infty (\boldsymbol{G}(P),\ \boldsymbol{N}(P))dF_P. \quad (3\cdot 9)$$

Now, carrying out the differentiation $\delta/\delta C_P$ on both sides, we obtain the required result (3·8).—

Using this theorem the solution of the equation (3·6) is found to be

$$f_X[C] = +\varepsilon\int_C (\boldsymbol{J}(P),\ \boldsymbol{N}(P))D_1(P-X)dF_P. \quad (3\cdot 10)$$

In (3·10) we have put the vanishing integration constant.

We have now to verify that the solution (3·10) satisfies the first relation of (3·5). This can be shown in the following manner:

First we substitute (3·10) into $[H_{\mathrm{I,II}}(P),\ f_X[C]]$:

$$\begin{aligned}&[H_{\mathrm{I,II}}(P),\ f_X[C]]\\ &= +\varepsilon^2\int_C [J^i(P),\ J^k(P')]\ A_i(P)N_k(P')D_1(P'-X)dF_{P'}. \quad (3\cdot 11)\end{aligned}$$

Since the point P as well as the point P' lie on C, and since C is space-like, one of the two points P and P' lie outside the light cone of the other. It follows, as one can verify by using the formula (2·11), that $[J^i(P),\ J^k(P')]$ has a non-vanishing value only when P' is an adjacent point of P. This situation enables us to replace $J^k(P'),\ N_k(P')$ in the integrand on the righthand side of (3·11) simply by $-J^{\bar{0}}(P')$ if one carries out the integration referring to the coordinate system in which the space-axes are tangent to C at P, (in this coordinate system the components of a vector are labeled by the barred suffixes) because in this reference system $\boldsymbol{N}(P')$ has the contravariant components (1, 0, 0, 0) at P' adjacent to P. Calculating (3·11) in this system we obtain

$$[H_{\mathrm{I,II}}(P),\ f_X[C]] = +\varepsilon^2\int_C [J^{\bar{i}}(P),\ J^{\bar{0}}(P')]A_{\bar{i}}(P)D_1(P'-X)dF_{P'},$$

which gives the vanishing result on account of the relation obtained by (2·11):

$$[J^{\bar{i}}(P), J^{\bar{0}}(P')] = \{\psi^*(P)\alpha\,\psi(P') - \psi^*(P')\alpha^{\bar{i}}\psi(P)\}\delta(\vec{x}_P - \vec{x}_{P'}) = 0.$$

In this way we have found the auxiliary condition which is compatible with the fundamental equation (III):

$$\left\{\begin{array}{l} \Xi_X[C]\Psi[C]=0 \\ \Xi_X[C]=\text{Div } \boldsymbol{A}(X)+\varepsilon\int_C(\boldsymbol{J}(P),\ \boldsymbol{N}(P))D_1(P-X)dF_P. \end{array}\right. \qquad \text{(IV)}$$

Since we have conditions of this form, one for each point X in the world, any one of these must commute with others. That this requirement is also satisfied can be verified as follows:

$$\begin{aligned} &[\text{Div } \boldsymbol{A}(X)+f_X[C],\ \text{Div } \boldsymbol{A}(X')+f_{X'}[C]] \\ &\quad=[\text{Div } \boldsymbol{A}(X),\ \text{Div } \boldsymbol{A}(X')]+[f_X[C],\ f_{X'}[C]]. \end{aligned}$$

The first term of the last expression vanishes because of $\Box D_1=0$. That the second term vanishes can be proved by carrying out the integration over P' in a reference system whose space axes are tangent to C at P.

§ 4. Gauge Transformation.

As mentioned in the preceding paragraph, only those Ψ's which satisfy the condition (IV) represent physically significant states of the system, so that only those linear operators which commute with $\Xi_P[C]$ represent physically measurable quantities. We shall show in this paragraph that these measurable quantities are so called gauge-invariant quantities which are invariant under the transformation

$$\left\{\begin{array}{l} \boldsymbol{A}\to \boldsymbol{A}+\text{Grad } \lambda \\ \psi\to e^{\frac{i\varepsilon}{\hbar}\lambda}\psi \\ \psi^*\to\psi^* e^{-\frac{i\varepsilon}{\hbar}\lambda} \end{array}\right. \qquad (4\cdot 1)$$

where λ is an arbitrary (real) scalar function of the world point satisfying

$$\Box\lambda=0.$$

We now introduce the quantity defined by

$$\left\{\begin{array}{l} \Gamma[C]=\Gamma_1[C]-\Gamma_2[C] \\ \Gamma_1[C]=\int_C\lambda(P')(\text{Grad}_{P'}\ \Xi_{P'}[C],\ \boldsymbol{N}(P'))dF_{P'} \\ \Gamma_2[C]=\int_C\Xi_{P'}[C](\text{Grad } \lambda(P'),\ \boldsymbol{N}(P'))dF_{P'}\,. \end{array}\right.$$

We shall now show that the quantity $\Gamma[C]$ has the following important properties:

$$[\Gamma[C],\ \mathbf{A}(P)]=\frac{\hbar}{i}\mathrm{Grad}\,\lambda(P) \qquad (4\cdot4)$$

$$[\Gamma[C],\ \psi_\rho(P)]=\varepsilon\lambda(P)\psi_\rho(P) \qquad (4\cdot5)$$

and

$$[\Gamma[C],\ \psi_\rho^*(P)]=-\varepsilon_\rho\psi^*\lambda(P). \qquad (4\cdot6)$$

The proof of (4·4) can be performed in the following way: First we have

$$[\Gamma_1[C],\ \mathbf{A}(P)]=\frac{\hbar}{i}\int_C\lambda(P')(\mathbf{N}(P'),\ \mathrm{Grad}_{P'})\ \mathrm{Grad}_{P'}\ D_1(P'-P)dF_{P'}.$$

We carry out the integration on the right-hand side, referring to the particular reference system in which the space axes are tangent to C at P. Then, since $\mathbf{N}(P')$ has contravariant components $(1, 0, 0, 0)$ at P' adjacent to P, we have here

$$(\mathbf{N}(P'),\ \mathrm{Grad}_{P'})\ \mathrm{Grad}_{P'}\ D_1(P'-P)=\frac{\partial}{\partial t_P}\mathrm{Grad}_{P'}\ D_1(P'-P),$$

so that, on account of the properties of the D-function stated in § 2,

$$[\Gamma_1[C],\ A_\mu(P)]=\begin{cases}0 & \text{for } \bar{\mu}=0\\ -\dfrac{\hbar}{i}\dfrac{\partial\lambda}{\partial x_{\bar{\mu}}} & \text{for } \bar{\mu}=1,\ 2,\ 3\end{cases}$$

Returning to the general reference system, this can be written as

$$[\Gamma_1[C],\ \mathbf{A}(P)]=-\frac{\hbar}{i}\Big\{\mathrm{Grad}\,\lambda(P)+(\mathbf{N}(P),\ \mathrm{Grad}\,\lambda(P))\ \mathbf{N}(P)\Big\}$$

A similar calculation can be carried out for the second term, with the result:

$$[\Gamma_2[C],\mathbf{A}(P)]=\frac{\hbar}{i}\int_C\mathrm{Grad}_{P'}\ D_1(P'-P)(\mathbf{N}(P'),\ \mathrm{Grad}_{P'}\ \lambda(P'))dF_{P'}$$

$$=-\frac{\hbar}{i}(\mathbf{N}(P),\ \mathrm{Grad}\,\lambda(P'))\mathbf{N}(P). \qquad (4\cdot8)$$

Now (4·7) and (4·8) together yield just the required relation (4·4).

The relations (4·5) and (4·6) are also easily proved; for the first term of Γ we have

$$[\Gamma_1[C],\ \psi_\rho(P)]=\varepsilon\int_C dF_{P'}\lambda(P')(\boldsymbol{N}(P'),\ \mathrm{Grad}_{P'})$$
$$\times\int_C -(W(P-P'')\alpha,\ \boldsymbol{N}(P''))_{\rho\tau}\psi_\tau(P'')D_1(P'-P')dF_{P''},$$

where we have applied the formula

$$[J^k(P'),\ \psi_\rho(P)]=-W_{\rho\sigma}(P-P')\alpha^k_{\sigma\tau}\psi_\tau(P').$$

Then, carrying out the integration with repect to $F_{P''}$ in the coordinate system, the space axes of which are tangent to C at P, thereby noticing $W_{\rho\sigma}(P-P')=\delta_{\rho\sigma}\delta(\vec{x}-\vec{x}')$ in this system, we find

$$[\Gamma_1[C],\ \psi_\rho(P)]=\varepsilon\int_C \lambda(P')(\boldsymbol{N}(P'),\ \mathrm{Grad}_{P'}\,D_1(P-P'))\psi_\rho(P)dF_{P'}$$
$$=-\varepsilon\lambda(P)\psi_\rho(P).$$

On the other hand, $\Gamma_2[C]$ contributes nothing to the commutator $[\Gamma[C],\ \psi_\rho(P)]$. This results from the fact that

$$[\Gamma_2[C],\ \psi_\rho(P)]=\varepsilon\int_C dF_{P'}(\mathrm{Grad}\ \lambda(P'),\ \boldsymbol{N}(P'))$$
$$\times\int_C -(W(P-P'')\alpha,\ \mathrm{N}(P'))_{\rho\tau}\psi_\tau(P'')D_1(P'-P')dF_{P''}$$
$$=-\varepsilon\int dF_{P'}(\mathrm{Grad}\ \lambda(P'),\ \boldsymbol{N}(P')).\ \psi_\rho(P)D_1(P-P').$$

Here appears in the integrand the function D_1 and not its derivative so that, when integrated over a space-like surface, the integral identically vanishes. Thus we have obtained (4·5). It goes without saying that the relation (4·6) can be verified in the same way.

The expressions (4·4), (4·5) and (4·6) mean that the quantity $\Gamma[C]$ can be regarded as the generating function of the gauge transformation (4·1). Namely, every dynamical quantity G transforms in this transformation according to

$$G(P)\rightarrow e^{\frac{i}{\hbar}\Gamma[C]}\,G(P)e^{\frac{i}{\hbar}\Gamma[C]} \qquad (4\cdot 9)$$

From (4·9) we can conclude that the necessary and sufficient condition for that G is a gauge-invariance is

$$[\Gamma[C], G(P)]=0, \qquad (4\cdot10)$$

or explicitly written

$$\int\lambda(P')([\mathrm{Grad}_{P'}, \Xi_{P'}[C], G(P)], \boldsymbol{N}(P'))dF_{P'}$$
$$-\int[\Xi_{P'}[C], G(P)](\mathrm{Grad}\,\lambda(P'), \boldsymbol{N}(P'))dF_{P'}=0. \qquad (4\cdot11)$$

Since λ satisfies (4·2) but otherwise may be chosen arbitrary, (4·11) is equivalent to

$$[\Xi_P[C], G(P)]=0. \qquad (4\cdot12)$$

We have thus obtained the required result: every gauge-invariant quantity commutes with Ξ, and *vice versa*.

Now, we must discuss how to introduce the gauge transformation into the frame of our theory. Such a discussion is necessary because the ordinary scheme of the gauge transformation in which the field quantities are transformed according to (4 · 1) can not be directly taken over into our theory. In fact, it can be seen at once that, if one adopts (4 · 1) and replaces the original field quantities by the quantities transformed according to (4·1) it destroys the invariance of various fundamental relations in our theory. The commutation relations, for instance, between new variables (which we distinguish by a prime) become then

$$\begin{cases} [A'^{\mu}(P), A'^{\nu}(P')]=\dfrac{\hbar}{i}g^{\mu\nu}D_1(P-P') \\ [\psi_\sigma'(P), \psi_\rho'^*(P')]_+=e^{\frac{i\varepsilon}{\hbar}\{\lambda(P)-\lambda(P')\}}W_{\sigma\rho}(P-P') \\ \qquad\qquad \text{etc.} \end{cases}$$

while the field equations for the free fields take the form

$$\begin{cases} \qquad \Box \mathrm{A}'=0 \\ \{(\alpha, \mathrm{Grad}-i\varepsilon/\hbar\,\mathrm{Grad}\,\lambda)+i\varkappa\beta\}\psi'=0 \\ \psi^{*\prime}\{(\alpha, \mathrm{Grad}+i\varepsilon/\hbar\,\lambda)-i\varkappa\beta\}=0, \end{cases}$$

and finally the generalized Schrödinger equation is

$$\left\{H' + \varepsilon((\psi^* \alpha \psi'),\ \mathrm{Grad}\,\lambda) + \frac{\hbar}{i}\frac{\delta}{\delta C_P}\right\}\Psi' = 0$$

with

$$H' = -\varepsilon(\psi^{*\prime}\alpha\psi',\ \boldsymbol{A}')$$

Thus our theory would not be invariant under gauge transformations.

But, as we shall show in the following, our theory is still invariant with respect to the change of the gauge; in other words, we can introduce the law of transformations for various quantites (in this case not only for the field quantities but also for the Ψ-vector $\Psi[C]$) in such a way that the fundamental relations maintain their forms when we go over from one gauge to another.

This can be done by decomposing the generating function $\Gamma[C]$ of the gauge transformation into two parts, one of which transforms the dynamical quantities, the other the Ψ-vector in such a way that the above-mentioned requirements are fulfilled. This possibility of separation in a relativistically invariant manner (so was the construction of our fundamental relations !) may be regarded as an indirect proof of the compatibility of the Lorentz and the gauge transformations.

Since $\left[\Xi,\ H+\frac{\hbar}{i}\frac{\delta}{\delta C_P}\right]=0$, it is obvious from (4·3) that

$$\left[\Gamma[C],\ H+\frac{\hbar}{i}\frac{\delta}{\delta C_P}\right]=0,$$

whence we get

$$e^{\frac{i}{\hbar}\Gamma}\left(H+\frac{\hbar}{i}\frac{\delta}{\delta C_P}\right)e^{-\frac{i}{\hbar}\Gamma}\Psi=0. \tag{4·13}$$

We introduce now:

$$\Gamma[C]=\gamma[C]+\gamma'[C] \tag{4·14}$$

with

$$\gamma[C]=-\int_C \lambda(P')(\mathrm{Grad}_{P'}\,\mathrm{Div}\,\boldsymbol{A}(P'),\ \boldsymbol{N}(P'))dF_{P'} \\ +\int_C \mathrm{Div}\,\boldsymbol{A}(P')(\mathrm{Grad}\,\lambda(P'),\ \boldsymbol{N}(P'))dF_{P'} \tag{4·15}$$

$$\gamma'[C]=-\varepsilon\int_C \lambda(P')dF_{P'}\{\boldsymbol{N}(P')\,\mathrm{Grad}_{P'}\int_C(\boldsymbol{J}(P''),\ \boldsymbol{N}(P''))D(P'-P')dF_{P''}\}$$

$$+\varepsilon\int_C(\mathrm{Grad}\ \lambda(P'),\ \boldsymbol{N}(P'))dF_{P'}\int_C(\boldsymbol{J}(P''),\ \boldsymbol{N}(P''))D(P'-P')dF_{P''}$$

$$=-\varepsilon\int_C \lambda(P')(\boldsymbol{N}(P')\boldsymbol{J}(P'))dF_{P'} \qquad (4\cdot16)$$

and prescribe the transformation rules for the dynamical quantities and the ψ-vector by means of

$$\begin{cases} G(P)\rightarrow e^{\frac{i}{\hbar}\gamma[C]}\,G(P)e^{-\frac{i}{\hbar}\gamma[C]}=G'(P) & \\ & \text{for any dynamical quantity;} \quad \text{(V)} \\ \Psi\rightarrow e^{-\frac{i}{\hbar}\gamma'[C]}\,\Psi=\Psi' & \\ & \text{for the } \Psi\text{-vector.} \end{cases}$$

The relation (V) gives rise to the following transformation for A, ψ, ψ^*:

$$\mathrm{A}\rightarrow\mathrm{A}+\mathrm{Grad}\ \lambda \qquad \left.\begin{array}{l}\psi\rightarrow\psi\\ \psi^*\rightarrow\psi^*\end{array}\right\}\ \text{(unchanged)} \qquad (4\cdot17)$$

as can be easily verified. This guarantees the invariance of the commutation relations and of the free-field equations.

Noticing that

$$e^{\frac{i}{\hbar}\gamma}\frac{\hbar}{i}\ \frac{\delta}{\delta C_P}e^{-\frac{i}{\hbar}\gamma}=\frac{\hbar}{i}\ \frac{\delta}{\delta C_P}$$

which follows from

$$\frac{\delta\gamma[C]}{\delta C_P}=\frac{\delta}{\delta C_P}\int^C \mathrm{Div}\ (\mathrm{Grad}\ \lambda\ \mathrm{Div}\ \boldsymbol{A}-\lambda\ \mathrm{Grad}\ \mathrm{Div}\ \boldsymbol{A})dV$$

$$=\Box\lambda.\ \mathrm{Div}\ \boldsymbol{A}-\lambda\Box\ \mathrm{Div}\ \boldsymbol{A}=0,$$

one obtains from (4·13) and (4·14), using $e^{\frac{i}{\hbar}(\gamma+\gamma')}=e^{\frac{i}{\hbar}\gamma}e^{\frac{i}{\hbar}\gamma'}$ which results from the fact that γ commutes with γ',

$$e^{\frac{i}{\hbar}\gamma'}\Big(e^{\frac{i}{\hbar}\gamma}H\,e^{\frac{i}{\hbar}\gamma}+\frac{\hbar}{i}\ \frac{\delta}{\delta C_P}\Big)e^{-\frac{i}{\hbar}\gamma'}\,\Psi=0$$

or, multiplying from the left by $e^{\frac{i}{\hbar}\gamma'}$ and using (V).

$$\Big(H'+\frac{\hbar}{i}\ \frac{\delta}{\delta C_P}\Big)\Psi'=0.$$

Thus the convariance of our generalized Schrödinger equation is also proved. It is obvious that, on account of (4·10) and (4·14), the expectation value of any physically measurable quantity maintains its value in the transformation (V). Thus the invariance of our theory under the transformation of the gauge has been proved.

It may be of some interest to check our assertion by means of a direct calculation. What we must verfy is that the equation $\left(e^{\frac{i}{\hbar}\gamma}He^{-\frac{i}{\hbar}\gamma}+\frac{\hbar}{i}\frac{\delta}{\delta C_P}\right)$ $\times e^{-\frac{i}{\hbar}\gamma'}\Psi=0$ does really hold. Since γ' commutes with H (see § 3), the first term gives

$$e^{\frac{i}{\hbar}\gamma}He^{-\frac{i}{\hbar}\gamma}e^{-\frac{i}{\hbar}\gamma'}\Psi=e^{-\frac{i}{\hbar}\gamma'}\left(H+\frac{i}{\hbar}[\gamma,\ H]\right)\Psi$$
$$=e^{-\frac{i}{\hbar}\gamma'}\{H-\varepsilon(\psi^*\alpha\psi,\ \text{Grad}\ \lambda)\}\Psi.$$

As for the second, we obtain the following results:

$$\frac{\hbar}{i}\frac{\delta}{\delta C_P}e^{-\frac{i}{\hbar}\gamma'}\Psi=e^{-\frac{i}{\hbar}\gamma'}\frac{\hbar}{i}\left(\frac{\delta}{\delta C_P}-\frac{\delta\gamma'[C]}{\delta C_P}\right)\Psi$$

From (4·16)

$$\frac{\delta\gamma'[C]}{\delta C_P}=-\varepsilon\ \text{Div}(\boldsymbol{J}(P)\lambda(P))=-\varepsilon(\boldsymbol{J}(P)\ \text{Grad}\ \lambda(P))=-\varepsilon(\psi^*\alpha\psi,\ \text{Grad}\ \lambda).$$

whence

$$\frac{\hbar}{i}\frac{\delta}{\delta C_P}e^{-\frac{i}{\hbar}\gamma'}\Psi=e^{-\frac{i}{\hbar}\gamma'}\left(\varepsilon(\psi^*\alpha\psi,\ \text{Grad}\ \lambda)+\frac{\hbar}{i}\frac{\delta}{\delta C_P}\right)\Psi.$$

Summing up the results and multiplying from the left by $e^{\frac{i}{\hbar}\gamma'}$ we arrive at

$$\left(H+\frac{\hbar}{i}\frac{\delta}{\delta C_P}\right)\Psi=0.$$

which is the original equation. (to be continued.)

Physics Departments,
Tokyo University and Tokyo Bunrika Daigaku.

Progress of Theoretical Physics Vol. II, No. 3, Jul.~Oct. 1947.

On the Canonical Transformation in Quantum Theory.*

Ryōyū UTIYAMA.

(Received Feb. 8, 1947)

Introduction.

The canonical transformation in quantum mechanics was considered by several authors and the correspondence between the treatments in quantum and classical mechanics was made clear by Dirac(1) and Jordan(2) especially. Further Dirac anticipated an existence of the generalized transformation function. Recently Prof. Yukawa pointed out again the importance of this conception, and Prof. Tomonaga(3) has shown that in the field theory the general transformation function can exist in some restricted meaning using some canonical transformation. Hence it will be worth while to formulate the canonical transformation and the perturbation method as its application in the quantum theory of fields. Here it should be mentioned that recently Dr. S. Watanabe(5) has treated the same problem independently.

§ 1. Point Mechanics.

In the first place we shall treat the problem classically. For the sake of simplicity, we consider a mass-point in one-dimensional space, and take $L(q\,\dot{q}\,t)$ as a Lagrangian, where q is the coordinate and a function of time t, and $q=\frac{dq}{dt}$. The equation of motion is given by

$$p=-\frac{\partial H}{\partial q}, \qquad \dot{q}=\frac{\partial H}{\partial p} \tag{1}$$

where p is canonical conjugate to q, $H=H(p, q)$ is the Hamiltonian corresponding to L.

Now consider a canonical transformation $(p, q) \rightarrow (P, Q)$ produced by a substitution function $W(q, Q, t)$, where (P, Q) are the new canonical variables. This is defined by

---

* This work was read at the Symposium of the National Research Council of Japan, on Nov. 18, 1944.

$$p=-\frac{\partial W}{\partial q}, \qquad P=\frac{\partial W}{\partial Q} \tag{2}$$

Further the new Hamiltonian is

$$K(P, Q, t)=H(p, q, t)-\frac{\partial W}{\partial t} \tag{3}$$

where the p, q in the right-hand side are to be considered as functions of P, Q defined by (2).

The Hamilton-Jacobi's equation corresponding to (1) is

$$H\left(\frac{\partial S}{\partial q}, q, t\right)+\frac{\partial S}{\partial t}=0. \tag{4}$$

Assuming the solution of (4) to be $S(q, Q, t)$ (Q is an arbitrary constant), we obtain a canonical transformation $(p, q) \rightarrow (P, Q)$, provided that $-S$ is taken as a substitution function.

Now we have a well-known solution of (4), i. e.

$$W=\int_t^T L(q, \dot{q}, t)dt=-S. \tag{5}$$

If we take as $q(t)$ in (5) that which gives the extreme value to the integral (5), then the canonical variables $p(t)$, $q(t)$ at the time t are transformed to $P(T)$, $Q(T)$ at the time T using this W as the substitution function.

In the second place we consider the problem quantum mechanically. Following Dirac(5)

$$(Q'|P|q')=\frac{\hbar}{i}\frac{\partial}{\partial Q'}(Q'|q')$$

$$(Q'|p|q')=-\frac{\hbar}{i}\frac{\partial}{\partial q'}(Q'|q').$$

In general we obtain the following relation

$$(Q'|a(q, Q)|q')=a(q'Q')(Q'|q'). \tag{6}$$

Putting

$$(Q'|q')=e^{\frac{i}{\hbar}W(q'Q')} \tag{7}$$

we obtain

$$\left.\begin{aligned}(Q'|\,P\,|q') &= \frac{\partial W(q'Q')}{\partial Q'}(Q'|q') \\ (Q'|\,p\,|\,q') &= -\frac{\partial W(q'Q')}{\partial q'}(Q'|q')\end{aligned}\right\} \tag{8}$$

Comparing (6) with (8), we obtain the following as q-number relations;

$$P=\frac{\partial W(qQ)}{\partial Q}, \quad p=-\frac{\partial W(qQ)}{\partial q} \tag{9}$$

provided that the arrangement of q, Q in W are well-ordered. From (9) we see that

$$e^{\frac{i}{\hbar}W(q,\,Q)}=U\,(q,\,Q) \tag{10}$$

is the transformation function corresponding to (q'-representation→Q'-representation).

Following the Heisenberg's picture, the state function is transformed by

$$\psi \rightarrow \psi' = U\psi$$

corresponding to (10), and any operator O is transformed by

$$O \rightarrow O' = U\,OU^{-1}.$$

Putting $O=p$ or $=q$, then

$$p \rightarrow P=U\,pU^{-1}, \qquad q \rightarrow q'=Q=U.qU^{-1}. \tag{11}$$

The proof for the equivalence of this with (9) is given by Jordan.⁽⁶⁾

Now we consider the perturbation theory as an application of the canonical transformation. We assume that the Hamiltonian can be written as follows;

$$H=H_0(p,\,q)+V(p,\,q,\,t) \tag{12}$$

and further that we know the solution $W=W(qQt)$, which satisfies the following equation,

$$-\frac{\partial W}{\partial t}+H_0\!\left(-\frac{\partial W}{\partial q},\,q\right)=0. \tag{13}$$

Using this W, and defining p and P to be equal to $-\partial W/\partial q$, and $\partial W/\partial Q$ respectively, then we obtain the following W as an operator

$$W=\int_0^t H_0(p,\,q)dt$$

if we insert the above defined $p(P,\,Q,\,t)$ and $q(P,\,Q,\,t)$ into (13). Here we take Heisenberg's description, so $p,\,q$ are functions of t, but H_0 does not contain t explicitly, hence H_0 is constant, so we obtain

$$W=H_0(p,\,q)t$$

where we can take $p,\,q$ at any instant for those in H_0. Therefore

$$U=e^{\frac{i}{\hbar}H_0(p,\,q)t} \tag{14}$$

Furthermore, assuming H_0 to be any rational integral function of $p,\,q$,

$$\begin{aligned}H_0(p,\,q)&=U\,H_0(p,\,q)U^{-1}\\&=H_0(U\,pU^{-1},\;U\,qU^{-1})=H_0(P,\,Q).\end{aligned}$$

Therefore we can rewrite (14) as follows

$$U=e^{\frac{i}{\hbar}H_0(PQ)t}. \tag{14$'$}$$

Using this as the transformation function, the total Hamiltonian is transformed as follows

$$H(pq)\rightarrow K(PQ)=U^{-1}H(p,\,q)U\,.$$

Taking notice for the fact that we have also the same relation with (3) in q-number theory, we obtain

$$\begin{aligned}K(P,\,Q,\,t)&=H_0\Big(-\frac{\partial W}{\partial q},\,q\Big)-\frac{\partial W}{\partial t}+V(p,\,q,\,t)\\&=V\{U^{-1}PU,\;UQU^{-1}\}=V'(P,\,Q,\,t).\end{aligned} \tag{15}$$

And the new equations of motion become

$$\left.\begin{aligned}\frac{\hbar}{i}\dot{P}&=[V'(P,\,Q,\,t),\,P]\\\frac{\hbar}{i}\dot{Q}&=[V'(P,\,Q,\,t),\,Q]\end{aligned}\right\} \tag{16}$$

And the Hamilton-Jacobi's equation in q-number theory is

$$\frac{\partial S}{\partial t}+V\left(\frac{\partial S}{\partial Q},\, Q,\, t\right)=0. \tag{17}$$

Now we take on the contrary the Schrödinger picture. Then the state function ψ is transformed by (14)

$$\psi \rightarrow \psi' = U\psi = \chi$$

and the Schrödinger equation

$$\frac{\hbar}{i}\frac{\partial\psi}{\partial t}+\{H_0(p,\, q)+V(p,\, q,\, t)\}\psi=0$$

becomes as follows;

$$\frac{\hbar}{i}U^{-1}\frac{\partial\chi}{\partial t}+VU^{-1}\chi=0$$

(where p, q are independent of t)

Now putting

$$P=UpU^{-1}, \qquad Q=UqU^{-1} \tag{18}$$

we can write the above equation as follows;

$$\frac{\hbar}{i}\frac{\partial\chi}{\partial t}+V'(P,\, Q,\, t)\chi=0. \tag{19}$$

Here P and Q cantain t explicitly and the variation of these with t are given by

$$\frac{\hbar}{i}\dot{P}=[H_0,\, P], \quad \frac{\hbar}{i}\dot{Q}=[H_0,\, Q] \tag{20}$$

It is very interesting to take notice for the distinction between (16) and (20).

Now we consider a transformation function $U'(t)$ which transform the Schrödinger function $\chi(0)$ at the time 0 to $\chi(t)$ at t, i. e.

$$\chi(0) \rightarrow U'(t)\chi(0)=\chi(t),$$

The unitary character of U' is obvious from the fact that the length of the vector χ is always unity. Then putting

$$U'(t)=e^{\frac{i}{\hbar}W(t)} \tag{21}$$

we can obtain a hermitian operator $W(t)$ from (21). Then, from (19) we obtain

$$\frac{\hbar}{i}\frac{\partial\chi(t)}{\partial t}=\frac{\partial W(t)}{\partial t}\chi(t)=-V(PQt)\chi(t)$$

or

$$\frac{\partial W}{\partial t}+V(PQt)=0.$$

Using the relation

$$P=\partial W/\partial Q$$

which is also satisfied in this case considering that W is a functional operator of $Q(0)$ and $Q(t)$, we obtain

$$\frac{\partial W}{\partial t}+V\left\{\frac{\partial W}{\partial Q}, Q, t\right\}=0 \tag{22}$$

This is the Hamilton-Jacobi's equation, and W defined by (21) is a solution of (22).

§ 2. Quantum theory of Fields.

Now we can generalize the above arguments into the quantum theory of fields. Consider the case where two fields described by the following canonical variables are interacting with each other ;

$$p^{\mathrm{I}}_{\gamma}(\vec{x},\ t) \qquad q^{\mathrm{I}}_{\gamma}(\vec{x},\ t) \qquad \gamma=1.2\ldots\ldots N$$

and

$$p^{\mathrm{II}}_{\alpha}(\vec{x},\ t) \qquad q^{\mathrm{II}}_{\alpha}(\vec{x},\ t) \qquad \alpha=12\ldots\ldots M$$

and the Hamiltonian of the total system is given by

$$\overline{H}=\int H d\vec{x}$$

$$H=H^{\mathrm{I}}(p^{\mathrm{I}},\ q^{\mathrm{I}})+H^{\mathrm{II}}(p^{\mathrm{II}},\ q^{\mathrm{II}})+V(p^{\mathrm{I}},\ q^{\mathrm{I}}\ ;\ p^{\mathrm{II}},\ q^{\mathrm{II}}). \tag{23}$$

In this paper we shall represent as usual the spatial integral of any quantity $A(x)$ by

$$\overline{A}=\int A(x)\,d\vec{x}.$$

Using the Heisenberg's description, the commutation relations of the field quantities at any instant are given by

$$\left.\begin{array}{l} [p_{\Upsilon}^{K}(\vec{x}t),\, q_{s}^{L}(\vec{x}',\, t)]=\frac{\hbar}{i}\delta_{\Upsilon s}\delta^{KL}\delta(\vec{x}-\vec{x}') \\ K\cdot L=\text{I or II} \end{array}\right\} \qquad (24)$$

All other commutators are equal to zero.

The equations of fields are

$$\left.\begin{array}{ll} \frac{\hbar}{i}\dot{p}_{\Upsilon}^{\mathrm{I}}=[\overline{H},\, p_{\Upsilon}^{\mathrm{I}}]_{-} & \frac{\hbar}{i}\dot{q}_{\Upsilon}^{\mathrm{I}}=[\overline{H},\, q]_{-} \\ \frac{\hbar}{i}\dot{p}_{\alpha}^{\mathrm{II}}=[\overline{H},\, p_{\alpha}^{\mathrm{II}}]_{-} & \frac{\hbar}{i}\dot{q}_{\alpha}^{\mathrm{II}}=[\overline{H},\, q] \end{array}\right\} \qquad (25)$$

Now we consider only Bose-statistics, then we can take the Schrödinger picture using the conception of the functional derivative. For the Fermi-statistics the above mentioned treatment is very difficult, so we does not consider the latter in this paper.

In the Schrödinger picture, p, q are to be considered as functional operators independent of t. Defining the functional derivative by the following;

$$\frac{\partial\overline{F}[y(x)]}{\partial y(x_P)}=\lim_{\substack{\varepsilon\to 0\\ V\to P}}\frac{F[y(x_P)+\varepsilon(x_P)]-F[y(x_P)]}{\int_V\varepsilon(x)dv}{}^{(7)},$$

(hereafter a functional F of any function $y(x)$ is expressed by $F=F[y(x)]$.) we consider the following equation analogous to Hamilton-Jacobi's equation in point mechanics,

$$-\frac{\partial\overline{W}[q^{\mathrm{I}}q^{\mathrm{II}},\, Q^{\mathrm{I}},\, Q^{\mathrm{II}},\, t]}{\partial t}+\int H^{\mathrm{I}}\left[-\frac{\partial\overline{W}}{\partial q_P^{\mathrm{I}}},\, q^{\mathrm{I}}(x_P)\right]dv_P$$

$$+\int H^{\mathrm{II}}\left[-\frac{\partial\overline{W}}{\partial q_P^{\mathrm{II}}},\, q^{\mathrm{II}}(x_P)\right]dv_P=0 \qquad (26)$$

Assuming that we know the solution W of (26), insert it into (26) again then we obtain as in the case of point mechanics

$$\begin{aligned}&\overline{W}[q^{\mathrm{I}}q^{\mathrm{II}}, Q^{\mathrm{I}}, Q^{\mathrm{II}}, t]\\ &\qquad = t\{H^{\mathrm{I}}[p^{\mathrm{I}}, q^{\mathrm{I}}]+H^{\mathrm{II}}[p^{\mathrm{II}}, q^{\mathrm{II}}]\},\end{aligned} \tag{27}$$

where we use the following relation

$$-\frac{\delta\overline{W}}{\delta q^{\mathrm{I}}_{P}}=p^{\mathrm{I}}(x_P), \qquad -\frac{\delta\overline{W}}{\delta q^{\mathrm{II}}_{P}}=p^{\mathrm{II}}(x_P) \tag{28}$$

and further Q^{I}, Q^{II} are to be considered as arbitrary functions.

The transformation function U, corresponding to (27) is

$$U=e^{\frac{i}{\hbar}\overline{W}}=e^{\frac{i}{\hbar}t\{\overline{H}[p^{\mathrm{I}}, q^{\mathrm{I}}]+\overline{H}^{\mathrm{II}}[p^{\mathrm{II}}, q^{\mathrm{II}}]\}} \tag{29}$$

Now we put

$$\left.\begin{aligned} p^{\mathrm{I}\,(\mathrm{II})}_{\gamma(\alpha)} &\rightarrow P^{\mathrm{I}\,(\mathrm{II})}_{\gamma(\alpha)}(x, \mathrm{t})=U p^{\mathrm{I}\,(\mathrm{II})}_{\gamma(\alpha)} U^{-1}\\ q^{\mathrm{I}\,(\mathrm{II})}_{\gamma(\alpha)} &\rightarrow Q^{\mathrm{I}\,(\mathrm{II})}_{\gamma(\alpha)}(x, t)=U q^{\mathrm{I}\,(\mathrm{II})}_{\gamma(\alpha)} U^{-1}\end{aligned}\right\} \tag{30}$$

The former is a definition but the latter is derivable from (27), (29) and the former. This is easily verified from the analogy of this with those in the point mechanics. Further $\dot{P}$ $\dot{Q}$ are given by

$$\left.\begin{aligned} \frac{\hbar}{i}\dot{P}^{\mathrm{I}\,(\mathrm{II})}_{\gamma(\alpha)} &= [\overline{H}^{\mathrm{I}(\mathrm{II})}[p^{\mathrm{I}(\mathrm{II})}, q^{\mathrm{I}(\mathrm{II})}], P^{\mathrm{I}\,(\mathrm{II})}_{\gamma(\alpha)}]_{-}\\ \frac{\hbar}{i}\dot{Q}^{\mathrm{I}\,(\mathrm{II})}_{\gamma(\alpha)} &= [\overline{H}^{\mathrm{I}(\mathrm{II})}[p^{\mathrm{I}(\mathrm{II})}, q^{\mathrm{I}(\mathrm{II})}], Q^{\mathrm{I}\,(\mathrm{II})}_{\gamma(\alpha)}]_{-}\end{aligned}\right\} \tag{31}$$

And from (31) and (24) we obtain the following

$$\left.\begin{aligned} [Q^{\mathrm{I}}_{\gamma}(\vec{x}, t),\ Q^{\mathrm{I}}_{\delta}(\vec{x}', t)]_{-} &= D^{\mathrm{I}}_{\gamma\delta}(\vec{x}-\vec{x}, t-t')\\ [Q^{\mathrm{II}}_{\alpha}(\vec{x}, t),\ Q^{\mathrm{II}}_{\beta}(\vec{x}', t)]_{-} &= D^{\mathrm{II}}_{\alpha\beta}(\vec{x}-\vec{x}', t-t')\\ [Q^{\mathrm{I}}_{\gamma}(\vec{x}, t),\ Q^{\mathrm{II}}_{\alpha}(\vec{x}', t')]_{-} &= D^{\mathrm{I}\,\mathrm{II}}(\vec{x}-\vec{x}', t-t')\end{aligned}\right\} \tag{32}$$

Now putting

$$U\psi=\chi,$$

then the Schrödinger equation becomes

$$\frac{\hbar}{i}\frac{\partial\chi}{\partial t}+U\bar{V}U^{-1}\chi=0 \tag{33}$$

The above argument is exactly the same one with those given by Praf. Dr. Tomonaga.(8)

Now we express those transformation function of χ in general with S, which transforms the diagonal representation of Q at $t=0$ into those at the time t.

Then

$$\chi(t)=S(t)\chi(0),$$

and from the unitary character of S we can put

$$S=e^{\frac{i}{\hbar}\eta},$$

where η is a hermition operator.

Then

$$\frac{\hbar}{i}\frac{\partial\chi}{\partial t}=-U\bar{V}U^{-1}\chi=\frac{\partial\bar{\eta}}{\partial t}\chi$$

i. e.

$$\frac{\partial\bar{\eta}}{\partial t}+U\bar{V}U^{-1}=0. \tag{34}$$

Using this η. the transformation

$$(P^{\mathrm{I(II)}}_{\Upsilon(\alpha)},\ Q^{\mathrm{I\,(II)}}_{\Upsilon(\alpha)})\rightarrow(P'^{\mathrm{I\,(II)}}_{\Upsilon(\alpha)},\ Q'^{\mathrm{I\,(II)}}_{\Upsilon(\alpha)})$$

can be expressed by

$$\frac{\delta\bar{\eta}}{\delta Q'^{\mathrm{I\,(II)}}_{\Upsilon(\alpha)}}=P'^{\mathrm{I\,(II)}}_{\Upsilon(\alpha)},\qquad \frac{\delta\bar{\eta}}{\delta Q^{\mathrm{I\,(II)}}_{\Upsilon(\alpha)}}=-P^{\mathrm{I(II)}}_{\Upsilon(\alpha)}. \tag{35}$$

Now defining

$$UVU^{-1}=V'(P'Q'), \tag{36}$$

then (34) is rewritten

$$\frac{\partial\bar{\eta}}{\partial t}+\bar{V}'\left[\frac{\delta\bar{\eta}}{\delta Q'^{\mathrm{I,II}}_{\Upsilon\cdot\alpha}},\ Q'^{\mathrm{I.\,II}}_{\Upsilon\cdot\alpha}\right]=0 \tag{34'}$$

This is the equation to be satisfied by η which is an analogue to classical Hamilton-Jacobi's equation.

Department of Physics,
Osaka University.

References

(1) P. A. M. Dirac: Phys. ZS. d. Sowjet. **3**, (1933), 64.
(2) P. Jordan: ZS. f. Phys. **38**, (1926), 513.
(3) S. Tomonaga: Progress of Theoretical Physics, **1**, (1946), 27.
(4) S. Watanabe: The lecture at the Symposium of the National Research Council of Japan, on Nov. 18, 1944.
(5) P. A. M. Dirac: cf. (1).
(6) P. Jordan: cf. (2).
(7) P. Jordan & W. Pauli: ZS. f. Phys. **47**, (1928), 151.
(8) S. Tomonaga: cf. (3).

Progress of Theoretical Physics Vol. II, No. 3, Jul.∼Oct. 1947.

Giant Stars Producing Energy by C–N Reactions.

Chûshirô Hayashi

(Received March 20, 1947)

§ 1. Introduction.

According to the shell source model of Gamow and Keller,(1) red giant stars are considered as being at the evolutional stages of the main sequence stars and generating energy by *C–N* reactions only in a shell inside which all the hydrogen contents have allready been consumed. However their results about the radii and luminosities of stars with the large mass are not definite, because the consistency between the luminosity and energy liberation is not taken into account.

In this paper possible structures of the existing giants which are generating energy by *C–N* cycles are looked into with the assumptions that (i) the chemical compositions are uniform in the envelope as well as in the core which contains no hydrogen, (ii) helium contents in the envelope are negligible, and (iii) the heavy elements constitute the Ressell mixture which contains certain franctions of *C* and *N*. As the typical giants, Capella and ζ Aurigae are taken up and equations of equilibrium are integrated inwards, considering the variations of the guillotine factor and of the mean molecular weight if necessary. When the solution of the envelope is of a centrally condensed type, near the centre the temperature and the density become high enough for *C–N* reactions to go on, and at a certain point their total energy liberation amounts to the luminosity. Inside the point all hydrogen are considered to have been converted into helium, so that energy sources are not present and the temperature is constant. The equations of the isothermal core have a solution being finite at the centre only when the hydrogen abundance in the envelope, which is equal to that of helium in the core, is limited to a certain value. In the case of too little hydrogen, the temperature rises inwards so slowly that the radius of the core is small as compared with its mass and the central density becomes infinite. On the other hand, for too much hydrogen the radius is too large and the mass vanishes before the centre is reached.

---

(1) G. Gamow and G. Keller, Rev. Mod. Phys. **17**, (1945) 125.

After integrations it is shown that both stars have solutions when their hydrogen are nearly 35 %, and the mass of cores are small as expected from the fact that the stars satisfy the Eddington's mass-luminosity relation approximately. It is also shown that the temperatures of the cores are very high, 42 and $65\cdot10^6$ deg. respectively, and electrons are degenerate near the centre where the densities are of the order 10^6 and 10^5 gr/cm^3. Thus these stars actually obey the Gamow's model, and all the giants which lie between the two in the L–R diagram can be considered as having the same structures and energy sources.

§ 2. Point Source Envelopes.

With negligible helium contents, the mean molecular weight μ is given by

$$\mu=\frac{1}{2X_H+\overline{n}_R(1-X_H)} \tag{1}$$

where X_H is the hydrogen abundance by weight, and $\overline{n}_R$, the number of free particles per unit atomic weight of the Russell mixture, is given in the Ströngren's table⁽²⁾ as the function of the temperature T and the ionization limit φ/kT. Let N_e be the electron density, then

$$e^{\varphi/kT}=2\frac{(2\pi mkT)^{3/2}}{h^3N_e} \tag{2}$$

where m is the electron mass. Values of $\overline{n}_R$ for $T<10^6$ and $\varphi/kT=10$, 15, 20, which are necessary for outer envelopes of ζ Aur., are supplemented by the same method as Ströngren.⁽²⁾ For $T>10^6$ and $\varphi/kT>4$, $\overline{n}_R=0.50$ can be taken within a small error. Opacity coefficients by absorption $\varkappa_t$, and by scattering $\varkappa_s$ are

$$\varkappa_t=3.9\cdot10^{25}\frac{1}{t}\frac{\rho}{T^{3.5}}(1-X_H^2) \tag{3}$$

$$\varkappa_s=0.19(1+X_H) \tag{4}$$

---

(2) B. Ströngren, Zs. f. Astrophys. 7, (1933) 22.

where ρ is the density and t is the guillotine factor. t and total opacity $\varkappa$ are tabulated by Ströngren⁽²⁾ and Morse⁽³⁾ except for $T < 4 \cdot 10^5$, where t can be put equal to unity because the low temperature region is quite small. When the temperature gradient becomes large as compared with the pressure gradient, which is the case where the central condensation of mass is relatively small, the convective equilibrium becomes stable in place of the radiative one. Then it is assumed that the temperature gradient is given by the formula of the adiabatic change of matter and radiation, with the ratio of specific heats of matter 5/3⁽⁴⁾

$$\left(\frac{1}{T}\frac{dT}{dr}\right)_c = \frac{\Gamma_2 - 1}{\Gamma_2}\frac{1}{P}\frac{dP}{dr} \qquad \frac{\Gamma_2 - 1}{\Gamma_2} = \frac{2(y+1)(y+4)}{5y^2+40y+32} \tag{5}$$

where P is the total pressure, $y = p_g/p_r$ ratio of gas and radiation pressure, and r distance from the centre. In the case of the radiative equilibrium where radiation flux at r is $L(r)/4\pi r^2$

$$\left(\frac{1}{T}\frac{dT}{dr}\right)_r = -\frac{3\varkappa\rho L(r)}{16\pi a c r^2 T^4} \tag{6}$$

The actual gradient is given by the smaller (absolute value) of the two.

The fundamental equations are, together with (5) or (6)

$$\frac{d}{dr}\left(\frac{k}{\mu H}\rho T + \frac{1}{3}aT^4\right) = -\frac{GM(r)\rho}{r^2}. \tag{7}$$

$$\frac{dM(r)}{dr} = 4\pi r^2 \rho \tag{8}$$

where $M(r)$ is the mass inside r, H proton mass, G the gravitation constant, and $L(r) = L$ until the energy generation begins. The observational data used as the boundary values are given in Table 1.

In the region where $M(r) \approx M$, the solutions are obtained analytically by the Ströngren's method.⁽⁵⁾ Using the values thus obtained, which are shown in Table 2, numerical integrations are started inwards. Taking $x = \log(R/r)$, $y = p_g/p_r$, $\log T$, and $\varphi = M(r)/M$ as variables, four figures calculations are performed with the Runge-Kutta's method, a step of x is 0.02

---

(3) P. M. Morse, Astrophys. Journ. **92**, (1940) 27.

(4) S. Chandrasekhar, An Introd. to the Study of Stellar Structure (1939).

(5) B. Ströngren, Zs. f. Ap. **2**, (1931) 345.

until $x=1$, henceforth 0.5 and it is about 70 steps to the cores in Cap., and more in ζ Aur.

Table 1.

star	spectral type	effective temperature	mass $M/M_\odot$	radius $R/R_\odot$	luminosity $L/L_\odot$
Cap. A	GO	5200	4.18	15.9	120
ζ Aur. K	cK4	3200	14.8	200.	6310

Table 2.

star	X_H	μ	starting value				interface			
			T	$\xi=r/R$	$M(r)/M$	$y=p_g/p_r$	T^*	$\log(r^*/R_\odot)$	$M^*/M_\odot$	$\log\rho^*$
Cap.	0.40	0.909	$6\cdot10^5$	0.6708	0.9926	18.62	$38.7\cdot10^6$	$\bar{2}.660$	0.286	1.51
	0.37	0.948	$6\cdot10^5$	0.6794	0.9925	17.95	$40.7\cdot10^6$	$\bar{2}.558$	0.233	1.51
	0.31	1.035	$6\cdot10^5$	0.6963	0.9922	16.73	$50.2\cdot10^6$	$\bar{2}.102$	0.100	1.48
ζ Aur.	0.43	0.874	$1\cdot10^5$	0.7310	0.9969	2.960	$49\cdot10^6$	$\bar{2}.797$	0.925	1.47
	0.37	0.948	$1\cdot10^5$	0.7520	0.9968	2.750	$64\cdot10^6$	$\bar{2}.281$	0.413	1.58
	0.31	1.035	$1\cdot10^5$	0.7720	0.9965	2.630	at $\xi=0.065$ mass vanishes			

§ 3. Energy Generation.

When T and ρ become high enough and the C–N reactions begin to go on rapidly, the variation of $L(r)$ must be considered, i. e.

$$\frac{dL(r)}{dr}=4\pi r^2\rho\,\varepsilon \tag{9}$$

$$\varepsilon=\varepsilon_0\rho X_H X_{C+N}\tau^2 e^{-\tau} \tag{10}$$

$$\tau=3\left(\frac{\pi^2 M e^4 Z^2}{2\hbar^2 kT}\right)^{1/3},\quad \log\,\varepsilon_0=23.55 \tag{11}$$

where X_{C+N}, combined abundance of C and N, is assumed to be 1% of the Russell mixture. The right part of Table 2. shows the values where $L(r)$ vanishes. At this point P, T and $M(r)$ are continuous with those of the interior, but ρ must satisfy the following conditions owing to the discontinuity of the chemical compositions.

$$\frac{\rho}{\mu}=\frac{\rho^*}{\mu^*}$$

$$\mu^*=\frac{1}{\frac{3}{4}X_{He}+\bar{n}_K(1-X_{He})}, \qquad X_{He}=X_H \tag{12}$$

where ρ and ρ^* are densities out and inside the interface respectively.

§ 4. Isothermal Cores.

The integrations must be advanced further to find a solution which has no singularity at the centre, but in place of them, following Gamow, fitting of the values above gained to non-singular solutions of isothermal cores are tested. Electron pressure and density are generally given by

$$P_e=\frac{8\pi}{3h^3}(2mkT)^{3/2}kTG_{3/2}(\psi,\,T), \quad G_{3/2}=\int_0^\infty\frac{\left(1+\frac{kT}{2mc^2}u\right)^{3/2}u^{3/2}du}{e^{-\psi+u}+1} \tag{13}$$

$$N_e=\frac{\rho}{\mu_e H}=\frac{4\pi}{h^3}(2mkT)^{3/2}G_{1/2}(\psi,\,T),$$

$$G_{1/2}=\int_0^\infty\frac{\left(1+\frac{kT}{mc^2}u\right)\left(1+\frac{kT}{2mc^2}u\right)^{1/2}u^{1/2}du}{e^{-\psi+u}+1} \tag{14}$$

and the pressure of the heavy particles is approximately

$$P_N=\frac{\rho}{\mu_N H}kT \tag{15}$$

where μ_e and μ_N are the mean molecular weights of electrons and ions respectively. The equations of the isothermal cores are reduced to

$$\frac{d\psi}{dr}=-\frac{M(r)\mu}{r^2}\frac{GH}{kT} \tag{16}$$

$$\frac{dM(r)}{dr}=G_{1/2}(\psi,\,T)r^2\mu_e\frac{16\pi^2H}{h^3}(2mkT)^{3/2} \tag{17}$$

$$\frac{1}{\mu}=\frac{1}{\mu_e}+\frac{\Delta}{\mu_N}, \quad \Delta=\left(\frac{1}{G_{1/2}}\frac{dG_{1/2}}{d\psi}+\frac{1}{\mu_e}\frac{d\mu_e}{d\psi}\right) \tag{18}$$

where $\triangle < 1$ and negligible when electrons are degenerate. In our case the relativistic effects are small. i.e. $\psi kT/mc^2 < 1$, then $G_{1/2}$ can be expanded as follows

$$G_{1/2}(\psi, T) = F_{1/2}(\psi) + \frac{5}{4}\frac{kT}{mc^2}F_{3/2}(\psi) + \frac{7}{32}\left(\frac{kT}{mc^2}\right)^2 F_{5/2}(\psi) + \cdots$$

$$F_k(\psi) = \int_0^\infty \frac{u^k du}{e^{-\psi+u}+1} \tag{19}$$

Values of $F_k(\psi)$, Fermi-Dirac functions, are tabulated by Stoner.(6) The estimation of μ_e is not easy. For simplicity, it is considered that for dense matter space is closely packed with ions of sizes given by Slater's semi-empirical rules, and their outer shells, which are stripped of owing to their overlapping, give free electrons. For two cases, pure Russell mixture and $X_H = 0.4$, relations between μ_e and ρ are calculated, as shown in Fig. 1. For low densities, the temperature ionization predominates, and at a intermadiate density μ_e has its maximum values. Equations of the cores with $T = 38.7 \cdot 10^6$ and $X_{He} = 0.4$ are integrated from the centre outwards for the cases, $\psi_c = 50$, 30, 20, 10 and 4. Using the solutions, $\rho^* - r^*$ curves for given core's mass M^*, i. e. fitting curves, can be drawn. They are slightly shifted to the left as compared with Gamow's owing to the larger values of μ_e at the transition densities. Electro-static corrections will further shift them in the same direction.

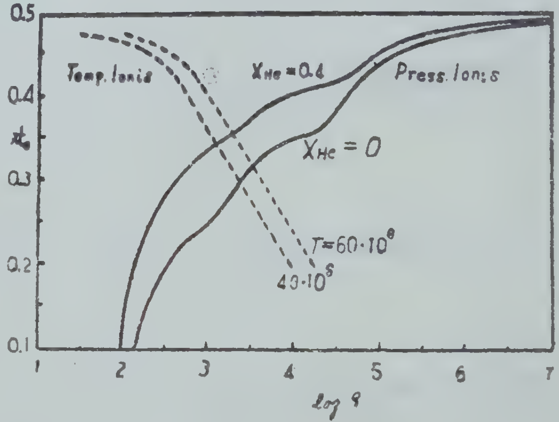


Fig. 1. Variation of μ_e

§ 5. Fitting of Cores and Envelopes.

In our case where relativistic effects and the variation of μ_e are small, following transformations of the fitting curves are permissible

---

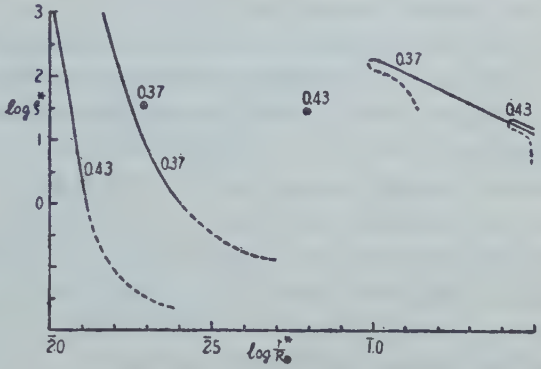
(6) E. C. Stoner, Phil. Trans. Roy. Soc. A, 237, (1938) 67.

$$M^*\mu^*T^{*-3/4}=\text{invariant.}$$

$$r^*\sim\frac{M^*\mu^*}{T^*}\sim\frac{1}{\mu^*T^{*1/4}},$$

$$\rho^*\sim\mu^*T^{*3/2}\sim M^{*2}\mu^{*5}\ (20)$$

Fitting curves for M^*, T^*, and μ^*, which are given in Table 2, are thus transformed from the original ones and they are compared with their corresponding points (ρ^*, r^*) of the envelopes, as shown in Fig. 2. For large X_H, M^* is also large and core curves are on the left of the points, however for small X_H the situations are reversed. By interpolation, fitting solutions are obtained, which are shown in Table 3.



ζ Aur. K

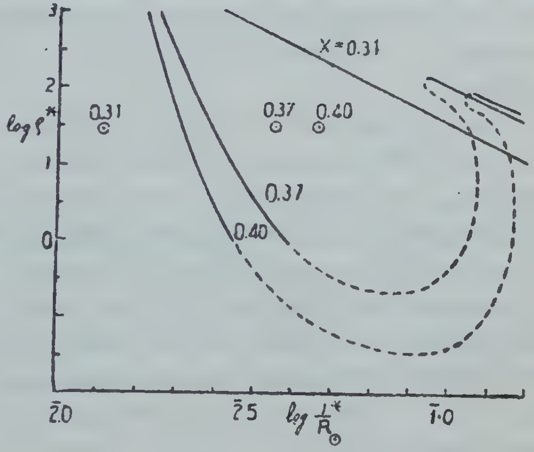


Fig. 2. Cap. A

Table 3.

star	X^*	T^*	$\log\rho^*$	$M^*/M_\odot$	$r^*/R_\odot$	ϕ_c	ρ_c
Cap.	0.35	$42\cdot10^6$	1.50	0.198	0.0302	17	$2.9\cdot10^5$
ζ Aur.	0.37	$65\cdot10^6$	1.58	0.410	0.0186	31	$1.2\cdot10^6$

§ 6. Conclusions.

In Cap. the region where 80% of the total energy is liberated is $r=2.57$ to $3.02\cdot10^{-2}R_\odot$. its mass $\Delta M=3.10^{-5}M$ and the mean energy generation is a order of 10^6 ergs/gr sec; the same is for ζ Aur. This is just what Gamow considers as a shell source. At $T=40\cdot10^6$ $C^{12}+H^1$ reaction is the slowest process, however for simplicity the rate of $N^{14}+H^1$ reactions is used above. At such a temperature reactions between hydrogen or helium and heavier nuclei can still be neglected. In the case of Cap., for $X_H>0.35$ envelopes are in radiative equilibrium everywhere, for $X_H<0.35$ convective currents appear near the interface, and the solution obtained lies on the boundary. In ζ Aur., for $X_H<0.43$ where integrations have been performed, convections always appear owing to the small values of y. and in the case of the solution $X_H=0.37$ they begin at $r=0.27\,R$. The accuracy of the solution for ζ Aur. is smaller than that for Cap. because of its low temperature near the surface, i.e. incomplete ionization, and the effects of viscosity to the adiabatic change of matter. And in Cap. X_H will have the uncertainty ±0.02 owing to uncertain factors of opacity t, energy generation ε, the variation of μ, and the electro-static effect for degenerate electrons.

Mass fractions of the cores are comparatively small, $M^*/M=0.047$ for Cap. and 0.028 for ζ Aur. When ε, i.e. radiative width Γ_γ or C–N abundance X_{C+N}, is decreased by a factor 10, they are increased to 0.056 and 0.037 respectively. In either case, they are smaller than that of the generalized Cowling's model. Then it may not be certain whether in the past the consumption of hydrogen in the central region was faster than the supplement by the circulation and diffusion, or they are not at the evolutional stages of the main sequence stars but were formed originally by the aggregation of matters around the nuclear stars which lacked hydrogen from the first.

In concluding the paper, the writer wishes to express his sincerest thanks to Professor H. Yukawa and Dr. S. Miyamoto for their continual encouragement and kind guidance during the work.

Institute of Cosmophysics
Kyoto University.

Progress of Theoretical Physics Vol. II, No. 3, Jul.～Oct. 1947.

Some Remarks on the Non-infinitesimal Commutation Relations.*

Eizo KANAI.

(Received April 10, 1947)

§ 1. Introduction.

As well known, the quantum theory of wave fields is made up of following two stages,

(1) quantum kinematics, (2) quantum mechanics.

The former is characterized by

(1′) commutation rules,

and the latter by

(2′) field equations.

At the present standpoint of view, (2′) are the same as in the classical theory and are derived from the Lagrangean function by the action principle. Then they are written as Poisson's equations, and are interpreted as q-number relations, when we pass to the quantum theory by means of (1′). Judging by this circumstance, we may rather see the essential part of the present theory in the stage of quantum kinematics (1), which represents the mutual interference of physical quantities.

But (1) cannot be defined willfully. They must satisfy at least following two conditions:

(a) compatibility with the field equations,

(b) relativistic invariance.

In fact, Heisenberg and Pauli⁽¹⁾ proposed a canonical procedure satisfying above two conditions automatically, but their method contained the so-called infinitesimal commutation rules. Indeed their quantization program showed the many successes in various directions, for example, in the explanation of the particle aspect of the wave field, but led to the divergence difficulties at the same time.

---

* The contents of this paper were read on Nov. 22, 1946, at the Symposium of the Japan Physical Society held at Kyoto University.

Since it has been often stated that the difficulties were due to the δ-function-like commutation relations between the fundamental quantities of the field, because they implied the eigenvalues of the density operator to be either 0 or ∞, that had to do with the point charactar of the particle associated with the field. Of course, we cannot exclude the possibility that these defects may be due to (2′), namely we may need to give up the simple linear field equations(2) or even the description of the natural phenomena by the differential equations.(3) However, we shall not touch this problem here and shall try to ask for the non-infinitesimal commutation rules, retaining the field equations in the present forms. At first sight, above two conditions (a) and (b) seem to permit many other possibilities than the present one. To take up this question and investigate the properties of such a newly quantized field is just the aim of this paper.

§ 2. The field equations and the general commutation relations.

Here we treat throughout the complex scalar field and represent its scalar wave variables by

$$\psi(x^{\mu}),\ \psi^{*}(x^{\mu})\quad (x^{\mu}=x, y, z, ct\,;\ x^{i}=x_{i},\ x^{4}=-x_{4})$$

which are Hermitian conjugate with each other, when considered as q-numbers. For simplicity, we assume the field equations as follows

$$\left(\frac{\partial^2}{\partial x^{\mu}\partial x_{\mu}}-\varkappa^2\right)\psi=0$$

or

$$\frac{\partial\psi}{\partial x^4}=\Pi,\quad \frac{\partial\Pi}{\partial x^4}=(\varDelta-\varkappa^2)\psi. \qquad (2\cdot 1)$$

Then, at the time $t=\text{const.}$, $\psi(\vec{x}, t)$, $\psi^{*}(\vec{x}, t)$, $\Pi(\vec{x}, t)$ and $\Pi^{*}(\vec{x}, t)$ are independent. So, in order to complete the quantum kinematics of § 1, (1), we must determine quantitatively the commutation relations between any pair of these variables. However, we take, for simplicity, following special selections

$$\left.\begin{aligned}
&[\psi(\vec{x},t),\ \psi^*(\vec{x}',t)]=f_1(\vec{x}-\vec{x}')\\
&[\Pi(\vec{x},t),\ \Pi^*(\vec{x}',t)]=f_2(\vec{x}-\vec{x}')\\
&[\Pi(\vec{x},t),\ \psi^*(\vec{x}',t)]=f_3(\vec{x}-\vec{x}')\\
&\text{other }[\quad]=0
\end{aligned}\right\}\qquad(2\cdot 2)$$

where we must decide the functions f_1, f_2, and f_3 by means of (a) and (b) of § 1. For the time being, we don't distinguish the $(+)$ type or the $(-)$ type of above brackets. But, for both types the next relations are true,

$$[A+B,\ C]=[A,\ C]+[B,\ C]$$

$$[A,\ B+C]=[A,\ B]+[A,\ C]$$

$$\frac{\partial}{\partial x^4}[A,\ B]=\left[\frac{\partial A}{\partial x^4},\ B\right]+\left[A,\ \frac{\partial B}{\partial x^4}\right]$$

$$[A,\ B]^*=[B^*,\ A^*]$$

§ 3. The postulate of (a).

For example, from (2·2) we have using (2·1)

$$0=\frac{\partial}{\partial x^4}[\psi,\ \psi^{*\prime}]=[\Pi,\ \psi^{*\prime}]+[\psi,\ \Pi^{*\prime}]=f_3(\vec{x}-\vec{x}')+f_3^*(\vec{x}'-\vec{x})$$

$$\therefore\quad f_3^*(-\vec{x})=-f_3(\vec{x})\qquad(3\cdot 1)$$

$$\begin{aligned}
0=\frac{\partial}{\partial x^4}[\Pi,\ \Pi^{*\prime}]&=(\Delta-\kappa^2)[\psi,\ \Pi^{*\prime}]+(\Delta'-\kappa^2)[\Pi,\ \psi^{*\prime}]\\
&=(\Delta-\kappa^2)f_3^*(\vec{x}'-\vec{x})+(\Delta'-\kappa^2)f_3(\vec{x}-\vec{x}')=0
\end{aligned}$$

$$\begin{aligned}
0&=\frac{\partial}{\partial x^4}[\Pi,\ \psi^{*\prime}]=(\Delta-\kappa^2)[\psi,\ \psi^{*\prime}]+[\Pi,\ \Pi^{*\prime}]\\
&=(\Delta-\kappa^2)f_1(\vec{x}-\vec{x}')+f_2(\vec{x}-\vec{x}')
\end{aligned}$$

$$\therefore\quad f_2(\vec{x})=-(\Delta-\kappa^2)f_1(\vec{x})\qquad(3\cdot 2)$$

And from the relation $[\psi,\ \psi^{*\prime}]^*=[\psi',\ \psi^*]$

$$f_1^*(-\vec{x})=f_1(\vec{x})\qquad(3\cdot 3)$$

Thus f_2 is uniquely determined by f_1, which is subjected by the condition

(3·3) as likely as f_3 by (3·1).

§ 4. The postulate of (b).

In the infinitesimal Lorentz-transformation of the space- and time-coordinates

$$'x^\mu = x^\mu + \varepsilon S^\mu{}_\nu x^\nu \cdot \qquad (S^{\mu\nu} = -S^{\nu\mu})$$

the field quantity $\psi(x)$ is transformed as follows

$$'\psi('x) = \psi(x).$$

Then

$$'\psi_\mu('x) \equiv \frac{\partial' \psi('x)}{\partial' x^\mu} = \frac{\partial \psi(x)}{\partial x^\nu} \cdot \frac{\partial x^\nu}{\partial' x^\mu} = \psi_\mu(x) - \varepsilon \psi_\nu(x) S^\nu_\mu .$$

Therefore

$$\left.\begin{aligned} &'\psi(x) = \psi(x) - \varepsilon \psi_\mu(x) S^\mu{}_\nu x^\nu \\ &'\Pi(x) = \Pi(x) - \varepsilon \Pi_\mu(x) S^\mu{}_\nu x^\nu - \varepsilon \psi_i(x) S^i_4 . \end{aligned}\right\} \qquad (4 \cdot 1)$$

Using these expressions, we have for example

$$\begin{aligned} ['\psi(x), '\psi^*(x')] = {} & [\psi(x), \psi^*(x')] \\ & - \varepsilon S^\mu{}_\nu (x^\nu [\psi_\mu(x), \psi^*(x')] + x'^\nu [\psi(x), \psi_\mu{}^*(x')]). \end{aligned}$$

Owing to the postulate of (b)

$$S^{\mu\nu}(x_\nu [\psi_\mu(x), \psi^*(x')] + x_\nu' [\psi(x), \psi_\mu{}^*(x')]) = 0.$$

Since it is necessary for this equation to be satisfied in any value of $S^{\mu\nu}$, the coefficients of $S^{\mu\nu}$ must be equal to zero. Firstly, putting the coefficients of S^{4k} equal to 0, we obtain,

$$\begin{aligned} 0 &= x^k [\Pi, \psi^{*\prime}] + x'^k [\psi, \Pi^{*\prime}] + x^4 [\psi_k, \psi^{*\prime}] + x^{4\prime} [\psi, \psi_k{}^{*\prime}] \\ &= x^k f_3(\vec{x} - \vec{x}') + x'^k f_3^*(\vec{x}' - \vec{x}) \\ &\qquad + x^4 \left(\frac{\partial}{\partial x^k} f_1(\vec{x} - \vec{x}') + \frac{\partial}{\partial x'^k}, f_1(\vec{x} - \vec{x}') \right) \\ &= (x^k - x'^k) f_3(\vec{x} - \vec{x}') \end{aligned}$$

$$\therefore \quad (x^k - x'^k) f_3(\vec{x} - \vec{x}') = 0. \qquad (k = 1, 2, 3)$$

Nextly, from the coefficients of S^{ik}

$$\begin{aligned} 0 &= x^k[\psi_i, \psi^{*\prime}] + x'^k[\psi, \psi_i^{*\prime}] - x^i[\psi_k, \psi^{*\prime}] - x'^i[\psi, \psi_k^{*\prime}] \\ &= x^k \frac{\partial}{\partial x^i} f_1(\vec{x} - \vec{x}') + x'^k \frac{\partial}{\partial x'^i} f_1(\vec{x} - \vec{x}') \\ &\qquad - x^i \frac{\partial}{\partial x^k} f_1(\vec{x} - \vec{x}') - x'^i \frac{\partial}{\partial x'^k} f_1(\vec{x} - \vec{x}') \\ &= (x^k - x'^k) \frac{\partial f_1}{\partial x^i} - (x^i - x'^i) \frac{\partial f_1}{\partial x^k} \end{aligned}$$

$$\therefore \quad (x^k - x'^k) \frac{\partial f_1}{\partial x^i} - (x^i - x'^i) \frac{\partial f_1}{\partial x^k} = 0 \quad (i,\ k = 1, 2, 3\,;\ i \neq k)$$

Applying the analogous procedure to other commutation relations in (2·2) and together with above equations, we can sum up the resulting conditions for f_1 and f_3 in next forms.

(1), with respect to f_1

$$\frac{\partial f_1(\vec{x})}{\partial x^i} + x^i(\Delta - \kappa^2) f_1(\vec{x}) = 0. \quad (i = 1, 2, 3) \tag{4·2}$$

(2), with respect to f_3

$$x^i f_3(\vec{x}) = 0. \qquad (i = 1, 2, 3) \tag{4·3}$$

(1) and (2) are entirely independent.

From (4·3) it is obvious that, as for f_3, there are no other possibilities than Dirac's δ-function. With this, and putting $f_1 = f_2 = 0$ we shall be led to the usually quantized field.

So, if we want to go beyond the infinitesimal quantization of the field, we should have $f_3 = 0$ and then investigate the solution of (4·2), to which we may add further condition

$$\lim_{|\vec{x}| \to \infty} f_1(\vec{x}) = 0. \tag{4·4}$$

Because physical quantities infinitely separated from each other are thought to have no relations.

§ 5. One solution of the equations (4. 2).

Slightly modifying eq. (4·2)

$$\frac{1}{x}\frac{\partial f_1}{\partial x}=\frac{1}{y}\frac{\partial f_1}{\partial y}=\frac{1}{z}\frac{\partial f_1}{\partial z}=(\kappa^2-\Delta)f_1$$

or
$$2\frac{\partial f_1}{\partial x^2}=2\frac{\partial f_1}{\partial y^2}=2\frac{\partial f_1}{\partial z^2}=(\kappa^2-\Delta)f_1.$$

To solve these partial differential equations, we assume

$$f_1=f(R),\quad R=x^2+y^2+z^2. \tag{5·1}$$

Then we obtain the ordinary differential equation to determine f

$$4Rf''+8f'-\kappa^2 f=0;\quad f'=\frac{df}{dR}.$$

Its solution can be written as follows

$$f=g\int_c e^{R\zeta+\frac{\kappa^2}{4\zeta}}\,d\zeta \tag{5·2}$$

where g is a constant proportional to $\hbar$, because the commutation rules should become zero as $\hbar$ tends to zero.

As for the integral path C in the ζ-plane, it must be selected to satisfy the condition

$$[\zeta^2 e^{R\zeta+\frac{\kappa^2}{4\zeta}}]_c=0 \tag{5·3}$$

as is easily verified. Two independent selections of this path give two independent solutions of (4·2) under the assumption (5·1).

For example, we can take a circle around the origin in the ζ-plane, which gives

$$f^{(1)}=\frac{\pi\kappa}{\sqrt{R}}gJ_1(i\kappa\sqrt{R}).$$

Or we can take the semi-real axis from $-\infty$ to 0, which gives

$$f^{(2)}=-\frac{\pi\kappa}{2\sqrt{R}}gH_1^{(1)}(i\kappa\sqrt{R}) \tag{5·4}$$

In the limit of large R, $f^{(1)}$ diverges strongly, while

$$f^{(2)} \approx \sqrt{\frac{\pi\kappa}{2}}\, g e^{-\kappa\sqrt{R}} / R^{3/4}. \qquad (5 \cdot 5)$$

So, on account of (4·4) we can only take the latter solution $f_1 = f^{(2)}(R)$, and then considering (3·3), g must be real number

$$g = \tau\hbar, \; \tau = \tau^*. \qquad (5 \cdot 6)$$

According to (5 · 5), f_1 is spread over the finite region, whose radius is about $1/\kappa$. Such function was once derived by Pauli,$^{(4)}$ who called it D_1.*

§ 6. Hamiltonian equations.

We have completed the quantum kinematics in § 3—§ 5, and hence the field equations (2 · 1) can now be interpreted as Heisenberg's equations of dynamical variables, which determine the changes in time of ψ and Π. Since their forms are of the differential equation, the development of ψ and Π must be the succession of infinitesimal transformations. Therefore, we can expect, in this system too, the presence of the Hamiltonian, that is, the operator translating the time infinitesimally.

In order to find this Hamiltonian, we assume the next expression

$$\overline{H} = i\hbar c \int \rho(\vec{x}', \vec{x}'')\left(\psi^*(\vec{x}')\Pi(\vec{x}'') - \Pi^*(\vec{x}'')\psi(\vec{x}')\right) d\vec{x}' d\vec{x}'' \qquad (6 \cdot 1)$$

where c-number function ρ should be decided to satisfy the equations;

$$i\hbar c \frac{\partial F}{\partial x^4} = F\overline{H} - \overline{H}F \qquad (6 \cdot 2)$$

for any quantity not containing the time explicitly. For example, putting $F = \psi$, we know a necessary condition for ρ

$$\int f_1(\vec{x} - \vec{x}')\rho(\vec{x}' - \vec{x}'')d\vec{x}' = \delta(\vec{x}, \vec{x}'') \qquad (6 \cdot 3)$$

Of course, ρ is not an ordinary function. However, if we consider ρ as a real and even improper function like the δ-function, we can easily show that (6·2) reproduces the field equations (2·1) and their conjugate ones for $F=\psi^*$, Π and Π^*. So (6·2) is true for all dynamical variables not containing the time explicitly. In other words, $\overline{H}$ is just the Hamiltonian of this system.

---

* Here I thank Prof. S. Tomonaga and Mr. T. Miyazima who kindly suggested to me the fact $f^{(2)} \propto D_1$.

The presence of $\overline{H}$ gives us the possibility of passing to the Schrödinger description of this field. Namely, representing the Schrödinger functional by $\overline{\Psi}$, Schrödinger equation becomes

$$i\hbar c\frac{\partial\overline{\Psi}}{\partial x^4}=\overline{H}\,\overline{\Psi}. \tag{6·4}$$

Or, in the stationary problem, we have

$$(\overline{H}-E)\overline{\Psi}=0. \tag{6·5}$$

where E is the energy of this state.

§ 7. The properties of the present field.

From (6·1) it is seen that $\overline{H}$ is Hermitian but not positive definite. These facts can also be shown by the following consideration. We begin with the introduction of new variables $\psi'(x)$ and $\Pi'(x)$;

$$\left.\begin{aligned}\psi'(\vec{x})&=\int\rho(\vec{x},\vec{x}')\psi^*(\vec{x}')d\vec{x}'\\ \Pi'(\vec{x})&=\int\rho(\vec{x},\vec{x}')\Pi^*(\vec{x}')d\vec{x}'\end{aligned}\right\} \tag{7·1}$$

which are different from $\psi^*(\vec{x})$ or $\Pi^*(\vec{x})$ respectively, merely by the normalization factor $\int\rho(\vec{x},\vec{x}')d\vec{x}'$, because of the δ-function-like charactor of ρ. Then we expand ψ, ψ', Π and Π' into Fourier series in the periodic volume V;

$$\left.\begin{aligned}&\psi(\vec{x})=\frac{1}{\sqrt{V}}\sum_{\vec{K}}a(\vec{K})e^{i(\vec{K},\vec{x})},\quad \psi'(\vec{x})=\frac{1}{\sqrt{V}}\sum_{\vec{K}}a'(\vec{K})e^{-i(\vec{K},\vec{x})}\\ &\Pi(\vec{x})=\frac{1}{\sqrt{V}}\sum_{\vec{K}}\omega_K b(\vec{K})e^{i(\vec{K},\vec{x})},\quad \Pi'(\vec{x})=\frac{1}{\sqrt{V}}\sum_{\vec{K}}\omega_K b'(\vec{K})e^{-i(\vec{K},\vec{x})}\\ &\qquad(\omega=\sqrt{K^2+\kappa^2}).\end{aligned}\right\} \tag{7·2}$$

Using the commutation relations between ψ, ψ', Π and Π';

$$\left.\begin{aligned} &[\psi(\vec{x}),\ \psi'(\vec{x}')]=\delta(\vec{x},\ \vec{x}') \\ &[\Pi(\vec{x}),\ \Pi'(\vec{x}')]=-(\Delta-\varkappa^2)\delta(\vec{x},\ \vec{x}') \\ &\text{other } [\quad]=0 \end{aligned}\right\} \qquad (7\cdot 3)$$

we obtain the commutation relations between the Fourier amplitudes

$$\left.\begin{aligned} &[a(\vec{K}),\ a'(\vec{L})]=\delta_{\vec{K},\vec{L}}=[b(\vec{K}),\ b'(\vec{L})] \\ &\text{other } [\quad]=0 \end{aligned}\right\} \qquad (7\cdot 4)$$

Or, performing another transformation;

$$\left.\begin{aligned} &A(\vec{K})=\frac{1}{\sqrt{2}}(a(\vec{K})+ib(\vec{K})),\ A'(\vec{K})=\frac{1}{\sqrt{2}}(a'(\vec{K})-ib'(\vec{K})) \\ &B(\vec{K})=\frac{1}{\sqrt{2}}(a(\vec{K})-ib(\vec{K})),\ B'(\vec{K})=\frac{1}{\sqrt{2}}(a'(\vec{K})+ib'(\vec{K})) \end{aligned}\right\} \qquad (7\cdot 5)$$

we have at last;

$$\left.\begin{aligned} &[A(\vec{K}),\ A'(\vec{L})]=\delta_{\vec{K},\vec{L}}=[B(\vec{K}),\ B'(\vec{L})] \\ &\text{other } [\quad]=0 \end{aligned}\right\} \qquad (7\cdot 6)$$

With these variables, $\bar{H}$ can be written as follows,

$$\begin{aligned} \bar{H}&=i\hbar c\sum_{\vec{K}}\omega_K(a'(\vec{K})b(\vec{K})-b'(\vec{K})a(\vec{K})) \\ &=\hbar c\sum_{\vec{K}}\omega_K(A'(\vec{K})A(\vec{K})-B'(\vec{K})B(\vec{K})). \end{aligned} \qquad (7\cdot 7)$$

$A(\vec{K})$ and $A'(\vec{K})$, $B(\vec{K})$ and $B'(\vec{K})$ are not Hermitian conjugate with each other, but their differences from the conjugate relations are only due to the constant normalization factors. So we get the same result as in the ordinary cases, that $A'(\vec{K})A(\vec{K})$ and $B'(\vec{K})B(\vec{K})$ are both Hermitian and have eigenvalues equal to zero or positive integer.

In this situation we are forced to adopt the $(+)$ type commutation relations in all above brackets. Because the $(-)$ type quantization permit the negative energy states even in the isolated system. In other words, according to present formalism, the scalar field must be quantized so as to obey the Fermi-statistics, which enables us to employ the Dirac's hole theory. But this fact contradicts the well known relation between the spin and the statistics, so long as we consider the spin in the sense of Pauli(4) and Fierz.(5)

Using the hole theory, we know that the present field consists of two kinds of particles, each of which has the energy $\hbar c\omega_K$ corresponding to the wave number vector $\vec{K}$.

On ther other hand, the appearance of two kinds of particles with the energy of opposite sign is sufficient to compensate the dynamical self-force in the interaction with other fields as is seen from the cases of Bopp(6) and Dirac.(7) However, we shall give up to extend the present method to the interacting systems, for it has already the clear discrepancy with the experience as for the relation between the spin and the statistics.

In conclusion, the author should like to express his deep gratitude to Professor H. Yukawa and Professor M. Kobayasi not only for their kind interest in this work but also for the constant inspiration derived from the many discussions at the institute. The author wishes to express his cordial thanks to Mr. S. Ozaki for his kind attention.

Institute of Theoretical Physics,
Kyoto University.

References

(1) W. Heisenberg and W. Pauli; Zs. f. Phys. **56** (1929), 1. Ibid. **59** (1930), 168.
(2) M. Born and L. Infeld; Proc. Roy. Soc. **144** (1934), 425.
(3) W. Heisenberg; Zs. f. Phys. **120** (1943), 513 and 673.
(4) W. Pauli; Solvay Berichite (1939).
(5) M. Fierz; Helv. Phys. Acta **12** (1939), 3.
(6) F. Bopp; Ann. d. Phys. **38** (1940), 345.
E. Kanai and S. Takagi; Prog. Theor. Phys. **1** (1946), 43.
(7) W. Pauli; Rev. Mod. Phys. **15** (1943), 175.

Progress of Theoretical Physics Vol. II, No. 1, Jul.～Oct. 1947.

The Theory of the Interaction of Elementary Particles.

I. The Method of the Theory of Elementary Particles.*

Shoichi SAKATA.

(Received April 12, 1947)

As is well known, the modern theory of elementary particles is composed of the following three stages. In the first place, a hypothesis concerning the model of the objects is postulated, with considerations as to what sort of elementary particles exist in nature, and what sort of interaction they are subject to. Next comes the stage in which the field quantum theory of Heisenberg and Pauli is applied to this system; and finally the process of drawing conclusions comparable with experiment by means of suitable approximation methods (chiefly perturbation theory). Therefore, when, as is the case today, these conclusions deviate markedly from experimental results (as does the life-time of the meson), or involve contradictions within themselves (for example, the infinite self-energy), the source of the difficulties must be sought for with careful retrospection as to the three points, corresponding to the above mentioned three stages: (i) the propriety of the model, (ii) the limit of the applicablity of quantum theory, (iii) the validity of the approximation method. This last-mentioned question of the approximation method especially perturbation theory, has been studied in detail by Tomonaga and Wentzel and their respective coworkers, with many fruitful results. Also, the works of Heitler, Wilson and Sokolow are in accordance with this line, and have succeeded in removing the divergenee difficulties in the high energy range. We have studied in particular the question of the model, and have shown that certain of the difficulties, for instance the problem of the lifetime of mesons, can be solved by an alternation of the model.(1) But as concern various divergence difficulties appearing in field theory such as that of the self-energy of an electron inherent ever since Lorentz electron theory, the question seems to be connected with the limit of applicability of quantum theory, and hence a new theory has long been awaited. It is well known that Heisenberg, discussing this problem, considers it as a contradiction arising from the over-looking of a new *universal constant* with the dimension of length, which ought to be introduced upon unifying the theory of relativity and quantum theory. This has become a guiding principle with reseachers of the theory of elementary particles today. We, too, trust that his anticipation will be confirmed in the future, but deem it necessary to dig down a little deeper in considering this problem.

---

* This article composes a part of the collaborated work on the theory of the interaction of elementary particles, by the *Elementary Particle Theory Group* of the Nagoya University. We with to express our cordial thanks to Dr. M. Taketani for his valuable discussions.

If the law of the development of physics, governing the direction in which theory would make progress in the future, could be discovered, and this law be applied consciously to the analysis of concrete problems, then we could surely attain our end with far greater facility and certainty than depending upon incidental successes due to the gifted intuition of researchers. It should be highly appraised that Heisenberg has endeavoured to discover such a law within the process of development of the theory of relativity and quantum theory by tracing the process historically. However, his arguments, which are based solely upon analogy, cannot be *propter hoc* though they may be *post hoc*. It cannot be deduced from the fact that the sun rises every morning, that it will rise again tomorrow. Thus Heisenberg himself does not anticipate an universal length with absolute confidence, but simply states that such a consideration lies close at hand.

Taketani states in his "*Formation of Newtonian Mechanics*"⁽²⁾ and other works that the following is the correct view. That is, we may look upon our cognition of Nature as a *dialectic logical* process proceeding spirally, repeating the circle corresponding to the "three stages of judgement" in Hegel's philosophy.⁽³⁾ He calls these the *phenomenological*, *substantialistic*, and *essentialistic* stages. The first, that is the phenomenological, stage implies the *an sich* stage in which separate facts are described, and corresponds, in Newtonian mechanics, to that stage represented by the role which the works of Tycho Brahe plays. The second is the stage in which the substantial structure embodying the occurence of phenomena is learnt, and the descriptions of the phenomena are put in order and are moulded into laws. Here law is significant as a property of the entity. This is the *für sich* stage where it is stated that a particular structure possesses such-and-such a phenomenon under such-and-such conditions, and corresponds to the Keplerian stage. Lastly, the essentialistic stage is said of the *an und für sich* stage in which it is made clear what phenomenon an entity with a given structure causes under given conditions, which is the Newtonian stage. Taketani applies this view to the present statues of meson theory and defines it to be in a state of searching for a path to sublimate itself from the substantialistic to the essentialistic stage, preliminarily setting things in order within substantialistic bounds.

Now, there are various theories postulated to cope with the contradictions in the present stage of the theory of elementary particles, most conspicuously appearing as the divergence difficulties, and we desire to find out the correct direction of development of the theory by classifying them from the above stand-point. Firstly, the cut-off method of Scherzer, March and others, the subtraction theories of Dirac, Bhabha, Heitler, Stueckelberg and others, and the λ-limiting process(4) of Wentzel and Dirac all attempt to remove the difficulty by subjecting the present theory to certain manipulations (which are relativistically invariant) in order to exclude the parts responsible for the divergences, and hence must be looked upon as belonging to the phenomenological stage. On the other hand, the new electro-magnetic field theory of Mie, Born and Bopp(5) attempts to clarify the substantial structure of the intertaction of elementary particles, so it may, together with Markov's theory which introduces non-commuting relations between the electromagnetic field quantities and the space co-ordinates of the electron, be classified in the substantialistic stage. It should, of course, be noted that such a classificatian must not be made merely formally.

According to the suggestion of Taketani in autumn 1945, we investigated in detail the structures of the Born-Bopp-sequence of theories which are in the substantialistic stage. Born's theory is famous for its non-linearity, and arouses much interest mathematically, but, as Bopp has pointed out, its essence does not lie in its non-linearity but in that its Lagrangian function involves derivatives higher than the second order of the potential. Bopp has in fact, succeeded in constructing a theory which, though linear, gives a finite self-energy. The quantization of this theory has been performed by Bopp himself but recently Podolsky and Kikuchi have also carried it out. With the same results as these, O. Hara(6) of our institute has proved, by the method of canonical transformations, that Bopp theory is a mixed field theory of the electro-magnetic field and the neutral vector Yukawa field with negative energy. It is termed a mixed field theory because it is of the same type as that in which Møller and Rosenfield considered, in order to remove the $1/r^3$ singularity of nuclear forces, a mixture of vector and pseudoscalar fields, and assumed particular numerical relations among the constants of interaction between them and the nucleons. As the static potential of an electro-magnetic field has the from $\frac{e^2}{r}$, while that of a negative energy neutral vector Yukawa field is $-g^2e^{-\varkappa r}/r$ (where e and g are the interaction constants of the respective fields, $h\varkappa/c$ the meson mass, and r the distance), the total static potential for a mixed field with $e=g$ become $e^2\left(\frac{1}{r}-\frac{e^{-\varkappa r}}{r}\right)$ and the self-energy takes a finite value. The original intention of Born and Bopp was a justification from the monistic standpoint, but we find it desirable, for the below-mentioned reason, to seek the essence of this theory in that it is a *mixed field* theory.

Hitherto, in field theory, the respective field interactions were taken out separately, and studied independently of others. But we wish to maintain that *it is the most effective method of developing the theory hereafter, and a promising path to the solution of the present difficulties, to investigate the internal correlations of the interactions between all the fields interacting with the same particle and all the particles acting as source of the same field.* The above-mentioned Bopp's theory and the mixed field theory of Møller and Rosenfeld certainly point to this direction. Also, the theory of Heitler and Ma, which assumes a multifariousness (various magnitudes of spin and charge) in the states of nucleons, which are the source of the nuclear field. and further Dirac's new quantization theory which leads to a mixture of positive and negative energy photon fields, both seem to verify the correctness of our standpoint. Dirac's electron theory started with the assumption that it was possible to evolve a relativistic quantum mechanics in connection with the one-body problem, but met with the negative energy difficulty, to cope with which it had, contradicting the first assumption, to pass over to the hole theory, which assumes the existence of infinitely many vacuum electrons. Is it not likely that the same applies to the divergence difficulty in the theory of elementary particles? We desire to look upon this divergence difficulty as indicating the narrowmindedness of the *formal logical* method hitherto employed in complete negligence of the correlations between elementary particles as a whole. The attitude of studying the various aspects of nature separately, apart from their relations as a whole was an effective method in the early stages of the development of science, and a fundamental condition for the great progress of modern science. As a result, we have become used to the way of thinking in which nature is regarded as an accidental collection of mutually isolated and independent objects capable of being considered separately from one another. But once this *formal logical* method, which originates in Bacon, exceeds its limits of applicability, it becomes a one-sided, narrow-minded method, and leads to insoluble contradictions. This is because we concentrate our attention on separate objects and neglect their correlations, on their existence and forget growth and annjhilation, on their stationary states and forget motion, seeing, as it were, only the trees and overlooking the forest. All the objects and phenomena occurring in nature are primarily correlated, depend on each other, and condition each other, so that if we take up any one phenomenon in an isolated form neglecting its correlation with environment, we are at once led to nonsensial results. The above-mentioned example of Dirac's electron theory is no special case, for whenever the *formal logical* method is applied inadequantely and carried through to the end, we arrive at a point diametrically opposite to the starting-point. There are presumebly very many confusions and contraditions in every branch of science, which arise because the majority of researchers are still not consciously aware of the limits of such a method. We desire to point out to the necessity, in the theory of elementary particles too, of being ever awake to the limit or applicability of this method, and taking the dialectic view point that all phenomena in the world are correlated with each other, and conditioned each other.

It is our intention, from this methodical view-point, to examine to what extent the difficulties of the theory of elementary particles may be removed by the mixed field theory. Also, we are investigating whether the methods of cut-off, subtraction, and λ-formalism which we mentioned above as being in the phenomenological stage may not, in some form or other, be corroborated by the mixed field theory.

However, such a theory as that of mixed fields, which merely postulates certain numerical relations between the interaction constants, is a theory in the phenomenological and substantialistic stages, involving an external and incidental cognition of the correlation between mutual interactions, and must, in future be elevated into the essentialistic stage, clarifying the more internal and inevitable correlations. Møller's 5-dimensional theory is noteworthy as a research tending in this direction, and we too must not forget that our final aim is to elevate our theory to a higher stage. In this case, most probably, as Taketani(7) has discussed in detail, the unifaction of the opposition between field and matter will play a fundamental role.* Also the existance of Heisenberg's "*universal length*" will be considered properly. The reason why we stress the necessity of a synthetical study of the mixed field theory including the multifariousness of the source is because we consider it an important work to set things in order within the phenomenological and substantialistic stages prior to passing on to the essentialistic stage. (to be continued)

Institute of Theoretical Physics,
Nagoya University.

References

(1) Sakata: The Problem of Models in the Theory of Elementary Particles. (Text to the Meson Discussion Meeting, 1943); Sakata & Inoue: Progress of Theo. Phys. **1** (1946) 143.
(2) Taketani: "Kagaku" **12** (1942), 807.
(3) Hegel: Wissenschaft der Logik (1812).
(4) Pauli: Rev. Mod. Phys. **15** (1943), 175.
(5) Bopp: Ann. d. Phys. **38** (1940), 345.
(6) Hara: Progress of Theo. Phys., in the press.
(7) Taketani: "Kagaku" **16** (1946), 199; Progress of Theo. Phys. **2** (1947), in the press.
(8) Wheeler & Feyman: Rev. Mod. Phys. **17** (1945), 157.

---

* Interesting in this connection is Wheeler's recent work.(8) Previous theories of elementary particles not only considered all forces to be conveyed by fields, but also treated matter itself as field. In short, they were "theories of fields". But he attempts to replace fields by interactions between matter, from the standpoint of action at a distance. This is a "theory of matter", in contrast to the previous "theories of fields". Further, he holds the view that the action of all matter existing in nature must be taken into account even when discussing the motion of a single electron, and thus makes clear the intrinsic nature of the reaction of fields in previous theories. This coincides with the dialectic view that the world must be interacted in correlation as a whole.

Progress of Theoretical Physics Vol. II, No. 3, Jul.~Oct. 1947.

On the Electromagnetic Properties of Nucleons, II.*

Eizo KANAI, Minoru KOBAYASI and Shigeo KOKAJI.

(Received May 1, 1947)

1. In the previous paper,(1) we have investigated the electromagnetic properties of nucleons by calculating the probabilities of transitions between two spin states of nucleons caused by radiation field, and obtained a finite result which may be interpreted as due to the electric dipole moment of the nucleon. But, we have not obtained the result corresponding to the magnetic moment of the nucleon within the order of approximation made in these calculations. Further, it seems to be very tedius to estimate this moment in the similar way by calculating the transition probabilities of higher processes.

In this paper, we shall attack this problem on the other side, that is, by estimating the energy of a nucleon placed in a statical electromagnetic field. Of course, it has been well known that the total energy of the system composed of a nucleon and nuclear and electromagnetic fields diverges as in the calculations of self-energies of elementary particles. And, this is just the reason why we have taken recourse to the transition probabilities in I. In connection with this circumstance, however, we may remember the fact that the same diverging results appeared in Dirac's positron theory and this difficulty was removed by getting rid of the diverging terms in a relativistic and gauge invariant way.(2) The diverging self-energy of electron has also been removed in Dirac's classical theory of electrons. In the case of Dirac's positron theory, the definite physical meanings such as the polarizability of vacuum were given to the remaining finite terms. Therefore, it may be expected that, if the diverging terms appearing in the calculation of the total energy of the system composed of a nucleon and statical electromagnetic field are removed in accordance with the physical demand as in the theory of positrons, the remaining terms may be interpreted physically by assuming they are due to the electromagnetic properties of the nucleon such as electric or magnetic moment or electromagnetic polarizabilities of the nucleon.

---

* The content of this paper was read at the annual meeting of the Japan Physical Society held on Apr. 29, 1946, at Tokyo University.

2. From the above view point, we shall first calculate the total energy of a nucleon placed in the constant electric or magnetic fields, then subtract the diverging terms from the results thus obtained following to Weisskopf's calculations in the theory of positrons.[(3)]

For simplicity, we assume here the pseudo-scalar meson theory and give the total Hamiltonian of the system as (1), (2a), (2b) and (2c) in I. We shall start from the case where no electromagnetic field is present, and repsesent the energy and wave function of the system in this case by W_i and $\Phi_i(g)$ respectively. We can never obtain the explicit expressions of them because of the divergence of self-energy of nucleon in the nuclear field. Nevertheless, we can put them formally in following forms:

$$W_i = E_i + \sum_n{}' \frac{H^g_{in} H^g_{ni}}{E_i - E_n} + O(g^4) \tag{1}$$

$$\begin{aligned} \Phi_i(g) = \Phi_i &+ \sum_n{}' \frac{H^g_{ni}}{E_i - E_n} \Phi_n \\ &+ \left\{ \sum_{n,m}{}'' \frac{H^g_{nm} H^g_{mi}}{(E_i - E_m)(E_i - E_n)} \Phi_n - \frac{1}{2} \sum_n{}' \frac{H^g_{in} H^g_{ni}}{(E_i - E_n)^2} \Phi_i \right\} + O(g^3) \end{aligned} \tag{2}$$

Here, H^g stands for H_{NM} in I. Next, we introduce the electromagnetic field represented by four potential $(A_0, \vec{A})$. Then, the wave function (2) turns into

$$\begin{aligned} \Phi_i(ge) = \Phi_i(g) &+ \sum_n{}' \frac{H^e_{ni}}{W_i - W_n} \Phi_n(g) \\ &+ \left\{ \sum_{n,m}{}'' \frac{H^e_{nm} H^e_{mi}}{(W_i - W_m)(W_i - W_n)} \Phi_n(g) - \frac{1}{2} \sum_n{}' \frac{H^e_{in} H^e_{ni}}{(W_i - W_n)^2} \Phi_i(g) \right\} + O(e^3) \end{aligned} \tag{3}$$

where $H^e = H_{ME} + H_{NME}$ in I. Consequently, the expression for the total energy is given by

$$\begin{aligned} [\Phi_i^*(ge),\ (H_0 + H^e)\Phi_i(ge)] &= (\Phi_i^*(g),\ H_0 \Phi_i(g)) + (\Phi_i^*(g),\ H^e \Phi_i(g)) \\ &+ \sum_n{}' \frac{(\Phi_i^*(g),\ H^e \Phi_n(g), (\Phi_n^*(g)),\ H^e \Phi_i(g))}{W_i - W_n} + O(e^3). \end{aligned} \tag{4}$$

The first term of this expression represents the energy of the system without electromagnetic field, and the second term is simply due to the presence of electromagnetic field without changing the state of the system. These terms have to be removed on the same ground as in Weisskopf's case. Further, when we expand the third term in power series of g, we get the first term which does not involve g and the second involving g^2 etc. Among these terms, those which do not involve g must be also removed as long as we start from the state where a proton or a neutron exists without meson field, because in such a case these terms are merely due to the zero-point oscillation of meson field, and according to Weisskopf the constant polarizability of vacuum is to be considered as zero. Thus, there remain the terms involving e^2g^2 in which we are first interested.

3. In order to simplify the problem, we first take up the case where only the electric field is introduced. Further, we put this field in the form

$$\vec{E}=\frac{1}{2}\vec{E}_0\,(e^{i\vec{\omega}\vec{x}}+e^{-i\vec{\omega}\vec{x}}),\qquad \vec{E}_0//\vec{\omega} \tag{5}$$

for the convenience of calculations, and finally in the limiting case where $|\omega|$ tends to zero we shall get the information of the case of constant field. By putting $\vec{E}$ in the form (5), we have the Fourier components of potentials in the next form:

$$A_0(\vec{k})=\begin{cases} i\,\dfrac{(\vec{\omega}\cdot\vec{E}_0)}{2\omega^2}=\dfrac{i}{2}\,\dfrac{E_0}{\omega}\,, & \vec{k}=\vec{\omega} \\[2ex] -i\,\dfrac{(\vec{\omega}\cdot\vec{E}_0)}{2\omega^2}=-\dfrac{i}{2}\,\dfrac{E_0}{\omega}\,, & \vec{k}=-\vec{\omega} \\[2ex] 0 & \vec{k}\neq\pm\vec{\omega} \end{cases}$$

$$\overrightarrow{A(k)}=0\,. \tag{6}$$

In the present case, we can put $H_{NMK}=0$, and for $\vec{r}_0=0$ ($\vec{r}_0$ is the position vector of the nucleon) we get

$$H^g=\frac{g}{\kappa^2}\sqrt{2\pi\mu^2c^4}\sum_{\vec{k}}\frac{(\vec{\sigma}\cdot\vec{k})}{\sqrt{\varepsilon_k}}\Big\{O^*(a_{\vec{k}}-b_{\vec{k}})+O(a_{\vec{k}}-\overset{*}{b}_{\vec{k}})\Big\} \tag{7}$$

$$\begin{aligned} H^e=\frac{e}{2}\sum_{\vec{k},\vec{k}'}A_0(\vec{k}-\vec{k}')\Big\{&\sqrt{\frac{\varepsilon_k}{\varepsilon_{k'}}}(a_{\vec{k}}^*a_{\vec{k}'}+b_{\vec{k}}a_{\vec{k}'}-a_{\vec{k}}^*b_{\vec{k}'}^*-b_{\vec{k}}b_{\vec{k}'}^*) \\ &+\sqrt{\frac{\varepsilon_{k'}}{\varepsilon_k}}(a_{\vec{k}'}a_{\vec{k}}^*+b_{\vec{k}'}^*a_{\vec{k}}^*-a_{\vec{k}'}b_{\vec{k}}-b_{\vec{k}'}^*b_{\vec{k}})\Big\} \end{aligned} \tag{8}$$

where the symbols have the usual meanings. By making use of these matrix elements, we get, after elementary but rather confounded calculations, for the diagonal matrix element of the energy expression

$$-2\pi\left(\frac{cg}{\varkappa^2\hbar^2}\right)^2(2\pi\mu^2)\frac{(\vec{\omega}\cdot\vec{E}_0)^2}{\omega^4}\sum_{\vec{k}}\left\{\frac{\omega^2}{(\varkappa^2+k^2)^2}-\frac{4(\vec{k}\cdot\vec{\omega})^2}{(\varkappa^2+k^2)^3}+\frac{4k^2(\vec{k}\cdot\vec{\omega})^2}{(\varkappa^2+k^2)^4}\right\}+O(\omega)$$

$$=-2\pi\left(\frac{eg\mu}{\varkappa^2\hbar^2}\right)^2E_0^2\int_0^\infty\frac{4\pi k^2dk}{(2\pi)^3}\left\{\frac{1}{(\varkappa^2+k^2)^2}-\frac{4k^2}{3(\varkappa^2+k^2)^3}+\frac{4k^4}{3(\varkappa^2+k^2)^4}\right\}+O(\omega). \tag{9}$$

In this expression, $\sum_{\vec{k}}$ means the summation over all possible intermediate states denoted by the momenta of virtually emitted mesons. By performing the integral over $\vec{k}$—space, we get the finite result

$$-\frac{5}{24}\frac{1}{\varkappa}\left(\frac{eg\mu}{\varkappa^2\hbar^2}\right)^2E_0^2+O(\omega) \tag{10}$$

for the energy expression which involves the interaction constants in the form e^2g^2. Further, putting $|\omega|\rightarrow 0$, we get finally for the additive energy of the nucleon in the constant electric field E_0

$$W_{\vec{E}_0}=-\frac{5}{24}\left(\frac{e^2g^2}{\hbar c\varkappa^2}\right)\frac{E_0^2}{\mu c^2} \tag{11}$$

4. For the case of magnetic field, the interaction terms of the total Hamiltonian are given by

$$H^e=\frac{ie}{4\pi\varkappa^2\hbar c}\int(\vec{A}\cdot\psi^*\nabla\psi-\vec{A}\cdot\psi\nabla\psi^*)dv$$

$$H^{ge}=\frac{ige}{\varkappa^2\hbar c}\left\{0(\vec{\sigma}\cdot\vec{A})\psi-0^*(\vec{\sigma}\cdot\vec{A})\psi^*\right\} \tag{12}$$

so long as we are interested in terms which involve e^2. Their matrix elements are given by

$$H^e=-\frac{e\mu c^2}{2\varkappa}\sum_{\vec{k},\vec{k}'}\frac{1}{\sqrt{\varepsilon_k\varepsilon_{k'}}}\Big\{(a_{\vec{k}}^*a_{\vec{k}'}-b_{\vec{k}}a_{\vec{k}'}-a_{\vec{k}}^*b_{\vec{k}'}^*+b_{\vec{k}}b_{\vec{k}'})(\vec{A}(\vec{k}-\vec{k}')\cdot\vec{k}')$$

$$+(a_{\vec{k}}a_{\vec{k}'}^*-b_{\vec{k}}^*a_{\vec{k}'}^*-a_{\vec{k}}b_{\vec{k}'}+b_{\vec{k}}^*b_{\vec{k}'})(\vec{A}'(\vec{k}'-\vec{k})\cdot\vec{k}')\Big\} \tag{13}$$

$$H^{\varphi e}=-\frac{eg}{\varkappa}\sqrt{2\pi}\sum_{\vec{k}}\frac{1}{\sqrt{\varepsilon_k}}\left\{0^*(\vec{\sigma}\cdot\vec{A}(\vec{r}_0))(a_{\vec{k}}^*-b_{\vec{k}})+0(\vec{\sigma}\cdot\vec{A}(\vec{r}_0))(a_{\vec{k}}-b_{\vec{k}})\right\}. \tag{14}$$

By assuming the periodic field

$$\vec{H}=\frac{\vec{H}_0}{2}(e^{i\vec{\omega}\vec{x}}+e^{-i\vec{\omega}\vec{x}}),\quad \vec{H}_0\perp\vec{\omega} \tag{15}$$

or

$$\vec{A}(\vec{k})=\begin{cases} i\,\dfrac{[\vec{\omega}\cdot\vec{H}_0]}{2\omega^2}, & \vec{k}=\vec{\omega} \\ -i\,\dfrac{[\vec{\omega}\cdot\vec{H}_0]}{2\omega^2}, & \vec{k}=-\vec{\omega} \\ 0, & \vec{k}\neq\pm\vec{\omega} \end{cases} \tag{16}$$

as in the case of electric field, we can calculate the finite diagonal matrix element of the total system in this case, and obtain in the limiting case $|\vec{\omega}|\rightarrow 0$

$$W_{\vec{H}^0}=-\frac{11}{48}\left(\frac{e^2g^2}{\hbar c\varkappa^2}\right)\frac{H_0^2}{\mu c^2} \tag{17}$$

for the additive energy of the nucleon in the homogeneous magnetic field $\vec{H}_0$.

5. The above results seem to lead us to the conclusion that owing to its virtually dissociated meson field the nucleon has phenomenoiogically the electric polarizability

$$\alpha=-\frac{5}{12}\left(\frac{e^2}{\hbar c}\right)\left(\frac{g^2}{\varkappa^2}\right)(\mu c^2)^{-1} \tag{18}$$

and the magnetic polarizability

$$\beta=-\frac{11}{24}\left(\frac{e^2}{\hbar c}\right)\left(\frac{g}{\varkappa}\right)^2(\mu c^2)^{-1} \tag{19}$$

instead of the statical electric or magnetic moment. These results can readily comprehended by considering that the electric density or electric current of meson cloud bounded by the bare nucleon is properly symmetric regarding with the spin direction of the nucleon and the additive finite energies obtained in the above calculations are due to the deformation of this density or current distribution caused by the applied electric or magnetic field. This circumstance seems to be inconsistent with the usual interpretation of the surplus magnetic moment of the nucleon. In our calculations, the additive energy which depends on the spin direction of the nucleon and has hitherto been attributed to the magnetic moment of the nucleon has been removed throughly with the diverging self-energy of the nucleon in oder to satisfy the physically permissible assumption.

6. Next, we shall discuss our results comparing with those obtained in I. The electric moment (7) of I is the non-diagonal element of the interaction matrix and we may expect from the order of approximation that its square will appear again in the energy expressions. But, it is easily seen that this term has no contribution in the present calculations, because the expression of the electric dipole moment implises the vector product of two direcrions of polarization of photons which should vanish in the present case. Therfore, we can conclude that the additive finite energies obtained in this paper arise from the other terms of the interaction matrix. It is desirable to find the non-diagonal matrix elements betweenin itial and intermediate or intermediate and final states which contribute to our results and decide whether electric or magnetic dipole should be attributed to the nucleon. But, judging from our preliminary calcucltions it seems hopeless to destinguish our matrix elements in this way because each separated term containing the spin operator in different manner diverges by itself. Therefore, we think we must be contented by obtaining finite result in each case. The terms electric and magnetic moments seem to be meaningless at least in such a case as in the system composed of the nucleon and electromagnetic and mesonic fields.

From above calculations, we can conclude at least that it is very doubtful to decide that the diverging additive energy of the nucleon placed in the homogeneous magnetic field should be attributed to its magnetic moment as in the usual way. Of course, our assumptions made in the subtraction method are, also, by no means free from ambiguity. But, generally speaking, it seems to us to be dangerous to draw out any conclusion from the calculations concerning the total energy of the system in the present stage of field theory. We think it is more plausible to discuss the properties of elementary particles by dealing with the transition processes where present theory gives the reasonable results so for as the first non-vanishing terms concern. From this point of view, we shall again treat the same problems in the next paper by calculating the transition probabilities of the nucleon caused by the statical inhomogeneous electromagnetic field, and compare those with the experimental results more directly.

In conclusion, we wish to express our thanks to Mr. S. Takagi for his kind assistance to our calcuations.

Institute of Theoretical Physics,
Kyoto University.

References.

(1) M. Kobayasi and E. Kanai: Prog. Theor. Phys. **1**. (1946) 21, In the following, this paper will be cited as I.

(2) P. A. M. Dirac: Proc. Cambr. Phil. Soc. **30**. (1934) 150; W. Heisenberg: Zs. f. Phys. **90**. (1934) 209; V. Weisskopf: Kgl. Danske Wid. Sels. Math-fys. Medd. **14**, No. 6 (1936).

(3) V. Weisskof: loc. cit.

Letters to the Editor

On the β-Ray Spectrum of Cu^{64}.

In this paper we suggest some modifications of Fermi theory of β-decay in order to explain the experimental energy spectrum of $C_u{}^{64}$, presented by Backus.(1) The fact that the ratio of the positrons and the electrons simultaneously emitted by $C_u{}^{64}$ are nearly independent of the energy of the particles at the low energy end, seems to be at variance with the Fermi theory where this ratio should be an increasing function of it. Moreover, the positron spectrum is in excellent agreement with the theory, so that this discrepancy can be attributed neither to scattering due to the finite thickness of the source nor to idiosyncrasy of the apparatus.

A theoretical investigation of this difficulty has been proposed by H. Lewis and D. Bohm.(2) They use as the interaction of β-decay the linear combinations of the five Fermi interactions, that is the difference between tensor and pseudoscalar, or the difference between scalar and vector interactions. These combined interactions essentially multiply Fermi distribution by $1-1/W$ for the electrons and $1+1/W$ for the positrons, W being the energy of the particles in units of mc^2. The results of this theory, though giving correctly the local form of the spectrum below 50 kev, could not be made to fit properly into the general form of the entire spectrum due to slow entailing properties of $1-1/W$.

As far as the light particles are concerned, the anomalous decrease of the electrons and the predicted behavior of the positrons when their energies are scanty, seems to suggest that something might prevent the light particles from closer approach to the nucleus. The influence of this effect should be masked by the Coulomb repulsion of the nucleus so that the positrons may be affected no appreciable change. We assume a repulsive potential of short range which is supposed to interact between the nucleon and the light particles;

$$V=(f^2/\hbar c)1/r\exp(-r/r_0) \qquad (1)$$

where r_0 ond f are the range and the strength of the repulsive force, and solve the Dirac equation of the electron and the positron in the Coulomb field of the final nucleus under the perturbation of this new potential. If we assume $f^2=\frac{e^2}{2}$, e being the elementary charge, we obtain the required ratio, without violating the general form of the entire spectrum.

As an alternative approach, we cut off the Coulomb potential at the surface, r_0 of the nucleus, inside which it may be a straight line (surface charge). Thus, the calculation goes analogously which gives the ratio

$$N_+/N_-=\text{const.}\exp(-2\pi aZW/p)$$
$$\times\left\{1+\frac{(aZ)^2\exp(2\pi aZW/p)}{2S^3(S+1)}\left(\frac{r_0}{r}\right)^{2S}\right\}^{-2} \qquad (2)$$

where Z is the atomic number of the nucleus, $a=1/137$, $p=\sqrt{W^2-1}$, $S=\sqrt{1-a^2Z^2}$, the bar indicates the mean over the entire domain.

If we put $\left(\frac{r_0}{r}\right)^{2S}\doteq\frac{1}{2}$, (2) modifies Fermi's result by means of the last factor of (2), so that N_+/N_- may be nearly independent of W. While our modifications give rise no appreciable correction on the spectrum above 200 kev, they decreases the total probability with a massive amount when Z becomes large; for instance, it decreases one twentieth when Z is 60. This corrections, with energy spectrum below 200 kev, are subject to direct experimental study. Since r_0 is approximately proportional to the main part of the domain, we consider that $\left(\frac{r_0}{r}\right)^{2S}$ is almost independent of r_0. It may be argued here that the conventional treatment of Fermi theory where the wave function of the electrons inside the nucleus is replaced by its value at the margin of the nucleus and the results depend on the nuclear radius, might be inconsistent.

Lastly we refer to the contributions of *C*-meson theory by Sakata⁽³⁾ or field theory by Pais⁽⁴⁾ to this problem, which has been developed to overcome the self energy difficulties. While the static forces acting to the electron derived from these theories have equal sign with the former of our theory, it is evident that the value of *f*-Charge, $f^2=2e^2$, which proves the cancellation of the divergent integrals, amounts 4 times larger than our value. If we accept the cut off procedure, we can show that *C*-meson or *f*-field will play no objectionable role, pue to smallness of their influences, within the limit of our perturbation treatment.

We are indebted to Prof. H. Yukawa, Prof. M. Kotani for the cordial interest to this problem. We also wish to express our heartly appreciations to Mr. M. Taketani for his advice and encouragement in this work.

S. Nakamura and K. Ono.

Tokyo University,
Physics Institute.
Aug. 1, 1947

References

(1) J. Backus, Phys. Rev. 68, (1945) 59.
(2) H. Lewis, and D. Bohm, Phys. Rev. 69, (1946) 12.
(3) S. Sakata, Kagaku, (1947) 579.
(4) A. Pais, Phys. Rev. 68, (1945) 227.

Modified Meson Pair Theory.

Exploring the possibility of explaining the whole phenomena about the nucleus and the cosmic ray on the basis of the meson pair theory, the author⁽¹⁾ introduced the process of exchanging the meson pair (Y^+, Y^0) or (Y^-, Y^0) between the nucleons, where Y^+, Y^- and Y^0 denote a positive, negative and neutral meson respectively. To account for the fact that the nuclear forces are nearly symmetrical with respect to the charge of the nucleon, it is of course necessary to preserve the process in the current meson pair theory, viz. the exchange of the pair (Y^+, Y^-). The mass μ_0 of the neutral meson must be assumed to be smaller than the mass μ_c of the charged meson in order to obtain a finite lifetime of the charged meson.

As the scalar theory can not give the right interaction between two nucleons, we investigated the case of spin 1/2.⁽²⁾ The requirement of "only one type of coupling" (among the five posstbilities) restricts the choice of the interaction to an unique posibility viz. the pseudovector interaction. With this interaction alone it is possible to explain qualitatively all known facts about nuclear two body systems. In these calculations the mesons with opposite charge were treated symmetrically and the neutral meson described by the Majorana's abbreviated theory. The divergent interaction between nucleons was avoided in a relativistically invariant way, by introducing an appropriate source function and taking the limit to a point source in the final results.

The dimensionless interaction constant G was roughly estimated to be $\sim 1/10$ by calculating the cross section for the nuclear scattering of the charged meson without changing their sign, though it may be determined more exactly from the cross section for the proton-proton scattering.

According to our theory, a charged meson can be absorbed by a nucleus with the emission of a neutral meson, provided the necessary energy is available. By a suitable choice of the mass diference $\mu_c - \mu_0$, we can make it possible that a *K*-orbit meson can be absorbed by Fe^{56} but not by C^{12}. In this way we can easily explain the experimental fact that the disintegration electrons are observed for negative mesons stopped in carbon but not for those stopped in iron.⁽³⁾ Considering the the energy of the disintegration electrons, we should take the mass difference as large as possible. Thus $\mu_c - \mu_0 \simeq 20$m (m denotes the electron mass) will be most probable. If this explanation is correct, He^4, Li^7, Be^9, B^{11} and A^{40} will belong in the class of C^{12}.

The productes of the spontaneous disintegration of a charged meson are, in the pair theory, a neutral meson, a neutrino and an electron or a positron. Assuming also the pseudovector interaction between the meson and the light particle, the life time of this meson decay was determined as a function of $\mu_c - \mu_0$.

The probability of the β-transformation through the intermediate state of mesons depends on the cutt-off distance r_0 for the potential between the nucleon and the light particle. If we assume that this process is the only one that causes the β-disintegration, and put $\mu_c - \mu_0 \sim 20$m, we must take $r_0 \sim$ one half the Compton wave length of the charged meson. By taking the value of r_0 to be about three times larger, the probability of the β-transformation becomes about one hundred times smaller. Thus we have another possibility of interpretation that the so-called allowed transition is caused by the direct interactson between the nucleon and the light particle as in the Fermi's original theory, and that the first forbidden transition takes place through the intermediate state of mesons. In this way we can explain simply the experimental facts that the first forbidden spectra of β-particles take also the allowed form.

The detailed calculations and discussions will be given in the later issue of this journal.

S. Noma.

Institute of Theoretical Physics,
Kyoto University.
Aug. 25, 1947.

References

(1) S. Noma, Progr. Theor. Phys. 1 (1946), 71.
(2) S. Noma, Prog. Theor. Phys. in Press.
(3) M. Conversi, E. Panchini and O. Piccioni, Phys. Rev. 71 (1947), 209.

Remarks on the Ferromagnetism of the Semi-conductors

A possible explanation of the ferromagnetism of the semi-conductors was proposed by S. Miyahara on the basis of Wilson s model of semi-conductors and of Slater's theory of the ferromagnetism. We try to treat the same problem by the statistical methods starting from Heitler-London model. We assume that each atom has two energy levels, ground level and excited level, and that at absolute zero temperature ground levels are fully occupied each by two electrons with antiparallel spins, while excited levels are all unoccupied. In the suitable temperature range one or two electrons in each ground level may be excited, and the remaining electrons of the singly excited atoms contribute to the ferromagnetism while the excited electrons do not.

Basing on this model, the statistical treatments are performed wholly analogous to the well-known two treatments for the order- and disorder-problem of alloys, namely, Bragg-Williams' theory and Bethe's theory.

1) In the treatment of Bragg-Williams'type, we assume the ordering force γM which make the spins of the remaining electrons of the singly excited atoms parallel and which we suppose, as in Weiss' theory of the ferromagnetism, to be proportional to the present magnetization M. From the simple consideration about the free energy of the system it can be shown that M is given by the root of following simultaneous equations similar to those of Weiss' theory:

$$y = \frac{kT}{\mu^2 \gamma N} x, \quad y = \frac{2(e^x - e^{-x})}{e^\lambda + e^{-\lambda} + 2(e^x + e^{-x})}$$

where $y = M/N\mu$ and $\lambda = U/kT$, and μ is the magnetic moment of an electron, N being the number of the total atoms.

2) In the treatment of Bethe's type, we consider the interaction of the atoms only between nearest neighbors and suppose that they are exchange type.

Therefore, we put the potential energies between neighboring atoms as follows:

$$V_{+-} = +J, \quad V_{++} = V_{--} = -J \quad (J > 0),$$

where V means the potential energy between both singly excited atoms, one has a remaining electron with + spin, the other with -spin, and so forth.

The potential energies of all other types of neighboring atoms are supposed to be zero. Following Bethe, we take up a typical atom which we shall call the central atom, and its Z-nearest neighbors which shall be called as the boundary atoms. We shall treat the interactions between them precisely and the influence of the other exterior atoms on the boundary atoms are taken into account in average such as it causes a tendency which makes the singly excited boundary atom to take the spin of the remaining electron +. Then the relative probability for the circumstance that the central atom is singly excited, the spin of the remaining electron being +, and that among the boundary atoms l atoms are doubly excited, m atoms are singly excited of which n atoms have the remaining electrons of + spin, is

$$P(+;\, l, m, n)=2\kappa\binom{Z}{l}(\kappa^2)^l\binom{Z-l}{m}(2\kappa)^m$$
$$\binom{m}{n}(\varepsilon\omega)^n(\varepsilon\omega)^{-(m-n)}$$

where $\kappa=e^{-U/kT}$, $\omega=e^{-J/kT}$ and ε is the factor which express the above denoted tendency due to the exterior atoms.

The relative probability for the circumstance that the central atom is singly excited and the remaining electron has + spin regardless of the situation of the boundary atoms is, thus

$$P(+)=\sum_{l=0}^{Z}\sum_{m=0}^{Z-l}\sum_{n=0}^{m}P(+;\, l, m, n)$$
$$=2\kappa[1+\kappa^2+2\kappa(\varepsilon\omega+\varepsilon^{-1}\omega^{-1})]^Z$$

All the other probabilities $P(-)$, $P(o)$, $P(\pm)$, the meaning of the notation being clear, are found easily in similar manner.

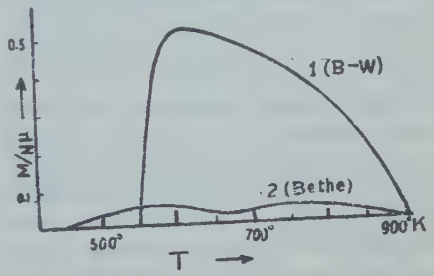
Now, in equilibrium, $Z[P(+)-P(-)]$ should be the difference of the average number of singly excited boundary atoms with remaing+spin and those with −spin. Hence we have

$$P(+)-P(-)=\frac{\varepsilon}{Z}\frac{\partial}{\partial\varepsilon}[P(+)+P(-)$$
$$+P(0)+P(\pm)]$$

from which we can determine ε. Finally $M/N\mu$ can be given by

$$\frac{M}{N\mu}=\frac{P(+)-P(-)}{P(+)+P(-)+P(o)+P(\pm)}$$

The relation between $M/N\mu$ and the temperature in both treatments are plotted for $Z=12$, $U\doteqdot\frac{1}{10}$ *e.v.*, $J\doteqdot\frac{1}{80}$ *e.v.* . The value of the constant γ is estimated by the relation $N\gamma\mu^2=ZJ$. We hope that the reason of the discrepancy of the results of two treatments will be understood by the more elaborate consideration.



It is our pleasure to express our thanks to Prof. M. Kobayasi for his constant encouragement. We are also indebted to Mr. E. Kanai for many valuable discussions.

T. Yamamoto and S. Takagi.

Institute of theoretical physics,
Kyoto University.
Oct. 25, 1947

On the Theory of Air Shower initiated by Secondary Soft Rays.

The theory of air showers has been initially developed on the assumption that they are large cascade showers initiated by primary electaons. But the proton-primary hypothesis has made it more probable that the air showers are produced by the tertiary soft rays, such as bremsstrahlung and decay electrons of mesons, the latter being produced by the primary protons on colliding with air nuclei.[1] But unfortunately the probabilities of the known processes in which a meson produces soft rays are all far too small to account for the observed frequency of air showers. One must thus assume some unknown soft-ray-producing processes of a meson to exist, or intyoduce a short-lived meson whose life is so short as of the order of 10^{-12} sec.[2]

In view of this situation it is of some interest to examine the possibility of explaining the air showers as initiated by soft rays (photons) produced directly by the primary protons (or eventually by secondary high-energy nucleons) when they collide with air nuclei and produce mesons.

When a primary proton of very high energy collides with an air nucleus it produces mesons with comparatively large probability. According to the theory of Heitler and others[3], mesons of rather low energies are likely produced, the incident proton changing thereby into a neutron. Thus a large velocity change of charge takes place in this collision, so that a photon (or eventually several photons) will be emitted. We can estimate the cross-section for this photon production by a simple consideration following Bloch and Nordsieck. We obtain thus

$$\sigma_{\text{photon}} dk = \sigma_{\text{meson}} \frac{1}{137\pi}\left(\frac{1}{u}\cdot\log\frac{1+u}{1-u}-2\right)\frac{dk}{k}$$

where σ_{meson} means the cross-section of the meson production by the high energy incident proton whose velocity is uc and k is the energy of the emitted photon.

Since u is almost unity for the protons responsible for the production of so hard photons which can initiate air showers, the logarithmic term can become so large as 15 ~30 for such high-energy protons. The cross section σ_{meson} is also found to be so large as of the order of 10^{-26} cm^2 so that our cross-section σ_{photon} is rather large.

Assuming this σ_{photon} we calculated the density-frequency relation of air showers and found that our cross-section is in fact large enough to be able to account for the observed frequency of air showers, if we may assume that σ_{meson} is of the order of $\pi(\hbar/\mu c)^2$, μ being the meson mass. The intensity of the primary protons in the high-energy region was thereby assumed to be a simple extrapolation from the low-energy region.

We also discussed the discrepancy of the observed density-frequency relations obtained by experiments using G–M counters[4] on one hand and ionization chambers on the other[5]. We concluded that this discrepancy shows the existence of heavily ionizing particles in air showers, which makes the results obtained by ionization chambers less reliable, because these particles apparently increse the frequency of dense bursts in the latter experiment. We estimated the amount of these particles and found that the existence of at most one heavily ionizing particle per 15 shower particles suffice to account for the discrepancy between these two kinds of experiments. Full account appears soon in Sc. Pap. I.P.C.R.

S. HAYAKAWA ane S. TOMONAGA.

Physics Institute, Central
Meteorlogical Observatory and
Tokyo Bunrika Daigaku.

Oct. 30, 1947

Reference

1) N. Hilbery: Phys. Rev., 60 (1941). 1.
2) M. M. Mill and R. F. Christy: Phys. Rev., 71 (1947), 275.
3) W. Heitler and P. Walsh: Rev. Mod. Phys., 17 (1945). 252.
4) G. Cocconi, A. Loverdo and V. Tongiorgi: Phys. Rev., 70 (1946). 841 and 846.
5) K. L. Kingshill and L. G. Lewis: Phys. Rev., 69 (1946). 159.

Progress of Theoretical Physics Vol. II, No. 4, Nov.~Dec., 1947.

On the Mesic Self-Field around a Nucleon*.

Kugao Nakabayasi.

(Received April 5, 1947)

Introduction.

As for models of the elementary particle there are at present two viewpoints, according as one regards it as a point singularity or as having a finite size.

If one adopts the second model, the self-field which the particle produces around itself is as a matter of course regular everywhere outside the particle. And no effect caused by its interaction with the field confronts with the difficulty of divergence. This holds in the quantum theory also; owing to the finite size of the particle, its field comes due to interference effect to contain no Fourier component belonging to arbitrary small wave-length. Thus the values of the interaction matrix-elements are cut down for momenta larger than a certain one determined by the particle radius, and thus one can get off from the difficulty of divergence. This procedure is sometimes called as the cut-off method. It was applied to the case of a nucleon interacting with meson field, classical as well as quantum theoretically, by many authors,(1) especially by Wentzel and by Tomonaga. However the cut-off momentum introduced is so to speak arbitrary in its magnitude, and if one take it large enough, the field energy becomes also much large in proportion. It influences decisively various effects and masks almost all effects coming from the well-established characters of the particle. Moreover in principle the inside of the particle, where the field equations fail to hold, comes into question, and in order to preserve the particle in a finite size some forces or a new cohesive field must be introduced. For otherweise the structure distribution may not be specified in a relativistically consistent way, and just by this reason one feels a difficulty for a relativistic formulation of the theory, both classical and quantum.

---

* Read at the Meeting of Sendai-Branch, the Physical Society of Japan, held on 1, March, 1947.

In the first model, where the particle is assumed as a point singularity, the field and its energy or generally conservative quantities become this times infinitely large in the neighbourhood of the particle. To avoid this difficulty, there are at least apparently two alternative ways. One of them is such that one modifies the field equations so as to make the field regular also at the vicinity of the particle, and was attempted for electron by Born and Infeld and Bopp.(2)

On the contrary, in the second alternative, one assumes that the field equations hold exactly up to the point of the particle, and modifies the definitions of the conservative quantities. Formally saying, the singular terms appeared therein are merely subtracted (the subtraction method); physically one may say that the point singularity contributes just the same singular terms with opposite signs, thus remaining conservative quantities finite and definite. Dirac and Pryce(3) showed for the case of electron interacting with electromagnetic field that in frame of the classical theory the above procedure can be in harmony with the conservation law and the relativity principle performed in a well-defined and self-consistent way. The field reaction on the electron was automatically introduced. Bhabha(4) applied this method to a nucleon interacting with neutral vector meson field, which much resembles electromagnetic field, and investigated especially the effect of the field reaction on the scattering of slow neutral mesons by a rest nucleon. Fierz(5) treated similar scattering problem for charged longitudinal vector meson.

As the subtraction method allows us to bring the field reaction into the theory, it is of great interest to investigate mesic self-field around a nucleon by this method. This is done in the following for the case of symmetric longitudinal vector meson, in frame of the classical theory. The classical equation of motion for the isotopic spin or the charged vector of a rest nucleon is deduced and the mesic self-field is analysed. We also determine the field energy which is free from any singularity and discuss the problem of proton-isobars.

The validity of the above classical treatment will require that the coupling is so strong that the field fluctuation can be neglected. However no crucial criterion for the validity is in hand, because the consequent quantum theory is not yet accomplished. As regards this point, however it comes into attention the quantum theory of meson scattering put forth by Heitler and by Wilson,(6) who have taken into account the effect of the field reaction in the first approximation of perturbation theory, subtracting therein also the infinite terms connected with the self-energy of the particle. It is not evident whether or not their procedure can be made relativistically consistent. If however it were some approximation of the consequent quantum theory of the subtraction method, the fact that their scattering cross-section is in agreement with Fierz's one seems to suggest that our classical calculation may be correct to some extent, even if not the coupling constant is so large.

§ 1. Field equations and their solutions associated with a nucleon.

The vector meson field is described by two complex conjugate four-potentials

$$U_\lambda \equiv \frac{1}{\sqrt{2}}(U_\lambda^{(1)}+iU_\lambda^{(2)}) \tag{1}$$

and $U_\lambda{}^*$ for the charged, and one real $U_\lambda^{(3)}$ for the neutral one. The field equations, with a nucleon as a source, are written as follows:

$$\frac{\partial U_{\lambda\mu}^{(l)}}{\partial x_\mu}+\kappa^2 U_\lambda^{(l)}=4\pi\left(\frac{1}{\sqrt{2}}g u_\lambda^{(l)}+\frac{1}{\sqrt{2}}f\frac{\partial u_{\lambda\mu}^{(l)}}{\partial x_\mu}\right) \tag{2}$$

with
$$U_{\lambda\mu}^{(l)}=\frac{\partial U_\mu^{(l)}}{\partial x_\lambda}-\frac{\partial U_\lambda^{(l)}}{\partial x_\mu} \tag{3}$$

κ stands for the reciprocal Compton wave-length of meson; $\frac{1}{\sqrt{2}}g$ and $\frac{1}{\sqrt{2}}f$ are the mesic charge and the dipole moment of the nucleon, resp. The source functions, $u_\lambda^{(l)}$ and $u_{\lambda\mu}^{(l)}$, contain the operators ρ_l which act on the nucleon state-functions involved. They form the isotopic spin of the nucleon, ρ, which has the same algebraic properties as the ordinary spin $\sigma=(\sigma_x, \sigma_y, \sigma_z)$. In particular, the third component ρ_3 has eigenvalues ±1, to which correspond proton and neutron states of the nucleon, respectively.

In the following we deal with the classical theory. The isotopic spin ρ is therein considered as a classical unit vector in the charge, say, as well as the spin σ in the ordinary space.

Suppose that the nucleon be rest at the origin. Since we assume the nucleon to be a point singularity, so its source density is conveniently represented by a three dinensional δ–function, $\delta(\boldsymbol{r})$.

With these assumptions, the field equations (2) turn out, after some reductions,

$$\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2}-\Delta+\varkappa^2\right)\varphi^{(l)}=4\pi\left(\frac{1}{\sqrt{2}}g\rho_l+\frac{1}{\sqrt{2}}\frac{g}{c^2\varkappa^2}\ddot{\rho}_l\right)\delta(\boldsymbol{r}), \tag{4a}$$

$$\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2}-\Delta+\varkappa^2\right)\boldsymbol{A}^{(l)}=4\pi\left(\frac{1}{\sqrt{2}}f[\rho_l\sigma,\ \mathrm{rot}]-\frac{1}{\sqrt{2}}g\frac{1}{c\varkappa^2}\rho_l\,\mathrm{grad}\right)\delta(\boldsymbol{r}). \tag{4b}$$

The terms which involve the time derivatives, $\ddot{\rho}_l$ and ρ_l, in the right hands, do appear from the fact that the charge of the nucleon can not conserve by itself only, that is, $\partial u_{\mu\lambda}/\partial x_\mu \neq 0$. For, the charged meson field also carries charge, and this implies the change of ρ_1 and ρ_2, which in turn affects ρ_3 due to the following condition for the charge vector:

$$\rho^2=1. \tag{5a}$$

We note for future use the following relations, which are obtained from (5a) by successive differentiations with respect to time:

$$\dot{\rho}\rho=0,\ \rho\ddot{\rho}+\dot{\rho}^2=0,\ \rho\dddot{\rho}+3\dot{\rho}\ddot{\rho}=0,\ \rho\ddddot{\rho}+4\dot{\rho}\dddot{\rho}+3\ddot{\rho}^2=0. \tag{5b}$$

Similar relations hold also for the spin vector σ.

In the above equations (4), the notations similar to the electromagnetic case are employed for the four-potentials, in terms of these the field strengths are written down as

$$\boldsymbol{E}^{(l)}=-\frac{1}{c}\boldsymbol{A}^{(l)}-\mathrm{grad}\,\varphi^{(l)},\quad \boldsymbol{H}^{(l)}=\mathrm{rot}\,\boldsymbol{A}^{(l)}. \tag{6}$$

Sometimes there appear the field quantities denoted by φ, $\boldsymbol{A}$, $\boldsymbol{E}$ and $\boldsymbol{H}$, which are in accordance with (1) defined by

$$\varphi=\frac{1}{\sqrt{2}}(\varphi^{(1)}+i\varphi^{(2)}),\quad \text{etc.}$$

The field equations (2) or (4) can be solved by means of the Green-function, if ρ and σ are known. In our case of the rest nucleon, the retarded solutions, only which suffice for later purpose, are given by

$$\varphi^{(l)}(\boldsymbol{r},\ t)=-\int_{-\infty}^{\tau_0} d\tau J_0(\varkappa s)\frac{d}{d\tau}\frac{1}{s_0}\left(\frac{1}{\sqrt{2}}g\rho_l+\frac{1}{\sqrt{2}}\frac{g}{\varkappa^2}\frac{d^2\rho_l}{d\tau^2}\right), \tag{7a}$$

$$\begin{aligned}\boldsymbol{A}^{(l)}(\boldsymbol{r},\ t)=-\int_{-\infty}^{\tau_0} d\tau J_0(\varkappa s)\frac{d}{d\tau}\frac{1}{s_0}\frac{d}{d\tau}\frac{1}{s_0}\Bigg(&\frac{1}{\sqrt{2}}f[\rho_l\sigma,\ \boldsymbol{r}]\\ &-\frac{1}{\sqrt{2}}\frac{g}{\varkappa^2}\frac{d\rho_l}{d\tau}\boldsymbol{r}\Bigg),\end{aligned} \tag{7b}$$

where $s=+\sqrt{s_0^2-r^2}$ and $s_0=-(ct-\tau)$; $t_0=\tau_0/c$ is the retarded time defined by $s^0=r$; J_0 is the Bessel-function of 0–th order. Integrations by parts yield the expressions which are convenient for small values of r. In the following we confine ourselves only to the case of longitudinal meson field, i. e., $f=0$, then we have for $\varkappa r\ll 1$

$$\sqrt{2}\,\varphi^{(l)}=\frac{1}{r}\left(g\rho_l+\frac{g}{\varkappa^2}\frac{d^2\rho_l}{d\tau^2}\right)-\frac{1}{2}g\frac{d\rho_l}{d\tau}-3g\int_{-\infty}^{\tau_0} d\tau\,\rho_l\frac{J_2(\varkappa s_0)}{s_0^2}, \tag{8a}$$

$$\begin{aligned}\sqrt{2}\,\boldsymbol{A}^{(l)}=&\frac{1}{r^2}\frac{g}{\varkappa^2}\frac{d\rho_l}{d\tau}\frac{\boldsymbol{r}}{r}+\frac{1}{r}\frac{g}{\varkappa^2}\frac{d^2\rho_l}{d\tau^2}\frac{\boldsymbol{r}}{r}-\frac{1}{2}g\frac{d\rho_l}{d\tau}\frac{\boldsymbol{r}}{r}\\ &+\frac{1}{8}r\varkappa^2 g\rho_l\frac{\boldsymbol{r}}{r}+rg\frac{\boldsymbol{r}}{r}\int_{-\infty}^{\tau_0} d\tau\,\rho_l\varkappa\frac{J_3(\varkappa s_0)}{s_0^2}.\end{aligned} \tag{8b}$$

Here all the terms preceding the integrals are to be taken at the retarded time, though it approaches to the time t as $r\rightarrow 0$. The same applies to the field strengths, which become by (6) and (8)

$$\begin{aligned}\sqrt{2}\,\boldsymbol{E}^{(l)}=&\frac{1}{r^2}g\rho_l\frac{\boldsymbol{r}}{r}+\frac{1}{r}g\frac{d\rho_l}{d\tau}\frac{\boldsymbol{r}}{r}-\frac{1}{2}\varkappa^2 g\rho_l\frac{\boldsymbol{r}}{r}\\ &+r\varkappa_2 g\frac{\boldsymbol{r}}{r^-}\int_{-\infty}^{\tau_0} d\tau\,\rho_l\frac{J_2(\varkappa s_0)}{s_0^2},\end{aligned} \tag{8c}$$

while $\boldsymbol{H}^{(l)}=0$. The expressions (8) are in turn correct to orders of r^0, r and r. Of course the integral terms are all regular for $r=0$.

§ 2. Conservative quantities and equation of motion for the charge vector.

Now the current density and the Poynting vector of the meson field are, as is well known, given by

$$\boldsymbol{j}=\frac{ic}{4\pi\hbar c}(\boldsymbol{E}^*\varphi-[\boldsymbol{H}^*,\ \boldsymbol{A}]-\text{compl. conj}\cdot), \tag{9a}$$

$$\boldsymbol{S}=\frac{c}{4\pi}\sum_{l}([\boldsymbol{E}^{(l)},\ \boldsymbol{H}^{(l)*}]+\varkappa^2\varphi^{(l)}\boldsymbol{A}^{(l)}). \tag{9b}$$

They are conservative so far as only the meson field is concerned. In the other words, they satisfy anywhere, except for the source point, the following continuity equations:

$$\rho+\text{div}\boldsymbol{j}=0, \quad H+\text{div}\,\boldsymbol{S}=0, \tag{10}$$

ρ and H being the charge and the energy density of the meson field, resp.

We now proceed to find the equation of motion for the charge vector. To do this, we calculate the charge and the energy carried out by the meson field through the surface of a small sphere with the nucleon as the center. These flux per unit time prove, using (8) and (9) and taking into account (5 ,

$$\int r^2 d\Omega j_r=-\frac{e}{2\hbar}\frac{g^2}{c^2\varkappa^2}\frac{d}{dt}\left(\frac{1}{r}[\rho,\ \rho]_3+\frac{1}{c}[\rho,\ \ddot{\rho}]_3\right)+e\frac{g^2}{\hbar c}[\rho,\ \boldsymbol{R}]_3, \tag{11a}$$

$$\int r^2 d\Omega\, S_r=-\frac{1}{2}\frac{g^2}{c^2\varkappa^2}\frac{d}{dt}\left(\frac{1}{2r}\dot{\rho}^2-\frac{1}{3c}\rho\dddot{\rho}\right)-\frac{g^2}{c}(\rho,\ \boldsymbol{R}), \tag{11b}$$

where

$$\boldsymbol{R}=\frac{3}{4}\rho+\frac{1}{2}\frac{1}{c^2\varkappa^2}\dddot{\rho}+\frac{3}{2}\int_{-\infty} dt'\rho(t')\frac{J_2(c\varkappa[t-t'])}{(t-t')^2}. \tag{12}$$

In the above, such terms which vanish as the radius of the sphere tends to zero are already omitted.

In order to guarantee the conservation laws for the total system, the nucleon plus meson field, the above flux must be both "perfect differential," or equal to the decrements of the conservative quantities contained inside the sphere. In particular, the infinite terms are themselves of forms of "perfect differential," and therefore allow us to "subtract" them by putting as

$$\varepsilon = \frac{1+\rho_3}{2} e + e \frac{1}{2} \frac{g^2}{\hbar c} \frac{1}{c\varkappa^2} \left(\frac{1}{r} [\rho, \boldsymbol{\rho}]_3 + \frac{1}{c} [\boldsymbol{\rho}, \rho]_3 \right), \tag{13a}$$

$$E = E' - \frac{1}{2} \frac{g^2}{c^2\varkappa^2} \left(\frac{1}{2r} \rho^2 - \frac{1}{3c} \rho\dddot{\rho} \right), \tag{13b}$$

where E' means the rest energy of the nucleon: $E' = Mc^2$.

The subtraction procedure means physically that the infinite term of the energy of the particle, for example, cancels out just the corresponding one inherent in the energy of the field. In this way we have a finite value not only for the energy of the total system, but also for the field energy. Now, in order to obtain the latter, we must determine the quantity to be subtracted from the usually defined field energy. This can easily done, after Pryce, l. c., by the requirement:

$$\overline{H}' = \lim_{r=0} \left\{ \int_r^\infty dv\, H - \int r^2 d\Omega\, K_r \right\}, \tag{14}$$

where H is the usual energy density of meson field:

$$H = \frac{1}{8\pi} \sum_l \left\{ \boldsymbol{E}^{(l)2} + \boldsymbol{H}^{(l)2} + \varkappa^2 (\varphi^{(l)2} + \boldsymbol{A}^{(l)2}) \right\} \tag{15}$$

The time derivative of K_r appeared already in the expression (13b). By calculations similar to for (11), it proves

$$\boldsymbol{K} = \frac{1}{16\pi} g^2 \frac{\boldsymbol{r}}{r^4} \left(1 + \frac{1}{c^2\varkappa^2} \rho^2 \right) \qquad \text{for small } r. \tag{16}$$

Here the terms involving $\log \varkappa r$ and r^0 are omitted, because they occur in forms multiplied by $\rho\rho$ and $\rho\rho$, resp., both being identically zero for the cases to be considered in below.

Returning to (13), we obtain, when combined with (11),

$$\rho_3 = -2 \frac{g^2}{\hbar c} [\rho, \boldsymbol{R}]_3 \tag{17a}$$

$$\frac{dE'}{dt}=0=\frac{g^2}{\hbar c}(\rho,\ \boldsymbol{R})\,. \tag{18}$$

They give the equation of motion for the third component of the charge vector under a constraint.

There lack yet the equations for the other components; to obtain them, we combine the condition (18) with the first of (5a). For instance, we eliminate ρ_2 and get

$$\rho_1[\dot{\rho},\ \boldsymbol{R}]_3=\rho_3[\rho,\ \boldsymbol{R}]_1\,,$$

which gives in view of (17a)

$$\rho_1=-2\frac{g^2}{\hbar c}[\rho,\ \boldsymbol{R}]_1 \tag{17b}$$

Similarly for ρ_2. They, together with (17a), can be conveniently written in a form of vector equation:

$$\boldsymbol{\rho}=-2\frac{g^2}{\hbar c}[\rho,\ \boldsymbol{R}]\,. \tag{19}$$

This is our fundamental equation for the charge vector of the nucleon. In some cases it is rather convenient to deal with $\rho=\frac{1}{2}(\rho_1+i\rho_2)$, which obeys

$$\rho=i2\frac{g^2}{\hbar c}(\rho R_3-\rho_3 R) \qquad \text{with} \quad R=\frac{1}{2}(R_1+iR_2) \tag{17c}$$

as is immediately seen from (19).

§ 3. Mesic self-field around a nucleon.

For each stationary motion of the charge vector, which fulfils (18) and (19), there exists a possible stationary meson field associated with the nucleon. It is necessarily bound only near the nucleon and from it far away fades out. The motion of the charge vector corresponding to such a self-field is expected to be in a state of motion with a circular frequency ω which is absolutely smaller than $c\varkappa$.

First we consider the ground state with $\omega=0$ or constant $\boldsymbol{\rho}$. It is in fact a possible solution of (18) and (19), for we have then for (12)

$$\boldsymbol{R}=\frac{2}{3}\rho c\varkappa\int_0^\infty ds\frac{J_2(s)}{s^2}=\frac{1}{2}c\varkappa\rho\,. \tag{20}$$

The self-field around the nucleon is given by (7) and becomes, noting $ct-\tau_0\doteqdot r$,

$$\varphi^{(l)}(\boldsymbol{r},\ t)=\frac{1}{\sqrt{2}}g\rho_1\varkappa\int_0^\infty ds\frac{sJ_0(s)}{(s^2+\varkappa^2r^2)^{3/2}}=\frac{1}{\sqrt{2}}g\rho_1\frac{1}{r}e^{-\varkappa r},\quad \boldsymbol{A}^{(l)}=0\,. \tag{21}$$

This corresponds to the static Coulomb-field of a electron and may be called as the ground mesic field. Owing to vanishing of vector potential, this field carries no charge and thus is composed of neutral mesons only. Hence the charge of the total system is merely $\frac{1+\rho_3}{2}e=e\frac{1}{2}(1+\cos\theta)$, where θ denotes the polar angle of ρ in the charge space.

Now, in the quantum theory, eigenvalues of ρ_3 are only ± 1, and therefore the total charge Q becomes $1\cdot e$ or $0\cdot e$. This shows that proton and neutron states with static meson field can exist, even if the effect of quantisation is taken into account. It is in marked contrast with charged theory where only charged mesons are assumed. In the latter theory, we are to put $R_3=0$, as is infered from (9a) and (11a), and therefore the equation (17c) can be satisfied only if $\rho_3=0$. Thus we have $Q=e/2$; but quantisation does never allow this to occur.

Next we proceed to study the exicited state. If the nucleon is to possess the constant charge, the charge vector ought to undergo a pure precessional motion about the third axis in the charge space. Thus the proper solution of (19) or (17 a, c) under the constraint (18) is expected to be

$$\rho_3=\cos\theta,\quad \rho=\frac{1}{2}\sin\theta\, e^{i\omega t}\quad (0<|\omega|<c\varkappa)\,. \tag{22}$$

To see this, we calculate $\boldsymbol{R}$ defined by (12), assuming (22), and obtain

$$R_3=\frac{1}{2}c\varkappa\rho_3\,,\quad R=\frac{1}{2}c\varkappa\left[1-\left(\frac{\omega}{c\varkappa}\right)^2\right]^{3/2}\rho\,. \tag{23}$$

We note that the integral occured in R becomes for $0\leqq|\omega|\leqq c\varkappa$

$$\int_{-\infty} dt'\, e^{i\omega t'} \frac{J_2(c\varkappa[t-t'])}{(t-t')^2} = c\varkappa\, e^{i\omega t} \int_0^{\infty} ds\, \frac{J_2(s)}{s^2} e^{-(i\omega/c\varkappa)s}$$

$$= c\varkappa\, e^{i\omega t} \frac{1}{3} \left\{ \left[1-\left(\frac{\omega}{c\varkappa}\right)^2\right]^{3/2} + i\left[\left(\frac{\omega}{c\varkappa}\right)^3 - \frac{3}{2}\frac{\omega}{ck}\right)\right\}$$

Thus the equation (17a) is satisfied identically, while the equation (17c) requires the following condition:

$$\frac{\omega}{c\varkappa} = \frac{g^2}{\hbar c}\cos\theta \left\{1-\left[1-\left(\frac{\omega}{c\varkappa}\right)^2\right]^{3/2}\right\} \tag{24}$$

This is the characteristic equation for ω, when θ is given.

The equation (24) has no solution with $|\omega| < ck$, unless

$$a \equiv \frac{g^2}{\hbar c}|\cos\theta| \geqq 2\cdot 3^{-3/4}\sqrt{2+\sqrt{3}} = 0{,}9785 \tag{25}$$

for any assigned value of θ. For the critical value of a, there exists just one solution, $|\omega| = \sqrt{\frac{\sqrt{3}}{2}}c\varkappa$; from that value of a, two possible $|\omega|$ appear, and split each other more and more until a approaches to 1. When a still more increases beyond 1, only the smaller one remains to exists and shifts unlimitedly to smaller values. (Cf. Fig. 1.) Thus for sufficiently strong coupling, the charge vector is almost rigidly fixed in the charge space, unless θ is equal or nearly equal to $\pi/2$. But the latter case can never occur, if the effect of quantisation is considered, because the charge vector is then exactly quantised in direction, as is shown in below, no matter magnitude of the coupling constant may be.

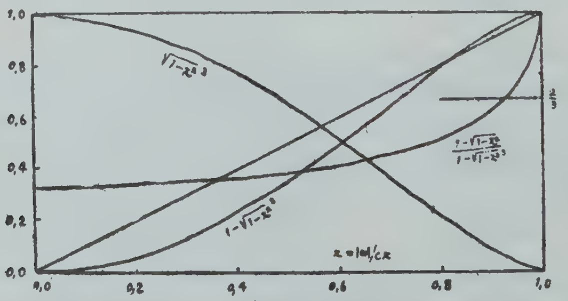


Fig. 1.

On the other hand, in the case of weak coupling: $g^2/\hbar c \ll 1$, no excited self-field can exist in the classical theory, so that the quantum fluctuation of field will play an important rôle.

So far we have only concerned with the case of symmetric theory. In the case of charged theory, we have to put $R_3=0$, as is mentioned above, and arrive at in place of (24)

$$\frac{\omega}{ck}=\frac{g^2}{\hbar c}\cos\theta\left\{-\left[1-\left(\frac{\omega}{c\varkappa}\right)^2\right]^{3/2}\right\}, \tag{24'}$$

in agreement with that given by Fierz, l. c. In marked contrast to the case of symmetric theory, this has always one solution (of course, except for $\theta\approx\pi/2$) for any value of coupling constant; it, so to speak, compensates to some extent the non-existence of the ground self-field.

The charged self-field for $|\omega|<c\varkappa$ proves, inserting (22) into (7) and (6),

$$\left.\begin{aligned}
\varphi&=-g\rho\left[1-\left(\frac{\omega}{c\varkappa}\right)^2\right]\frac{1}{\varkappa r}\,\frac{d}{dr}\int_0^\infty ds\frac{sJ_0(s)}{\sqrt{s^2+\varkappa^2r^2}}e^{-i\frac{\omega}{c\varkappa}\sqrt{s^2+\varkappa^2r^2}}\\
&=g\rho\left[1-\left(\frac{\omega}{c\varkappa}\right)^2\right]\frac{1}{r}e^{-\varkappa r\sqrt{1-\left(\frac{\omega}{c\varkappa}\right)^2}},\\
\boldsymbol{A}&=-\frac{1}{1-(\omega/c\varkappa)^2}\,\frac{1}{c\varkappa^2}\operatorname{grad}\varphi=g\rho\frac{1}{c\varkappa}\,\frac{\boldsymbol{r}}{r}\left\{\frac{1}{\varkappa r}+\sqrt{1-\left(\frac{\omega}{c\varkappa}\right)^2}\right\}e^{-\varkappa r\sqrt{1-\left(\frac{\omega}{c\varkappa}\right)^2}},\\
\boldsymbol{E}&=g\rho\varkappa\frac{\boldsymbol{r}}{r}\left\{\frac{1}{\varkappa r}+\sqrt{1-\left(\frac{\omega}{c\varkappa}\right)^2}\right\}e^{-\varkappa r\sqrt{1-\left(\frac{\omega}{c\varkappa}\right)^2}},\quad \boldsymbol{H}=0,
\end{aligned}\right\} \tag{26}$$

whereas the neutral self-field is the same as for the ground state. Thus the former reaches more far away than the latter, or has a longer range by the factor $\sqrt{1-\left(\frac{\omega}{c\varkappa}\right)^2}$.

§ 4. Charge vector and energy of the excited self-field.

The polar angle of the charge vector, θ, is determined if we fix the total charge Q. We now define the charge vector of the excited meson field, $\boldsymbol{L}$, so that the charge vector of the total syetem becomes $\boldsymbol{J}=\boldsymbol{L}+\frac{1}{2}\rho$. Its third component stands with the total charge in the relation:

$$Q=eL_3+e\frac{1}{2}(1+\rho_3). \tag{27}$$

From the conservation of Q, we get then $\dot{Q}=0=\dot{L}_3+\frac{1}{2}\dot{\rho}_3$, or in virtue of (17a)

$$\dot{L}_3=\frac{g^2}{\hbar c}[\rho,\ \boldsymbol{R}]_3, \tag{28a}$$

which leads to, by generalisation,

$$\dot{\boldsymbol{L}}=\frac{g^2}{\hbar c}[\rho,\ \boldsymbol{R}]. \tag{28b}$$

As the consequence of the equation of motion for ρ, J_1 and J_2 also turn out to be constants of motion.

The equation (28b) can be easily integrated for our case as follows: Since $\boldsymbol{J}=0$, $\dot{\boldsymbol{L}}$ points always towards the direction opposite to $\dot{\rho}$. Hence $\boldsymbol{L}$, together with ρ, undergoes a precession about the third axis in the charge space with the angular velocity $\boldsymbol{\omega}=\omega\boldsymbol{e}_3$. Moreover it is observed that $\dot{\rho}$ and $\boldsymbol{L}$ are both constant relative to the rotating system: ρ, $\dot{\rho}$, $[\rho,\ \dot{\rho}]$, and that $\boldsymbol{L}$ is proportional to $[\rho,\ \dot{\rho}]$, not only its direction but also in its time dependence. Thus we can put $\boldsymbol{L}=C[\rho,\ \dot{\rho}]$. The constant c is determined by the condition:

$$\dot{\boldsymbol{L}}=C[\rho,\ \ddot{\rho}]=\frac{g^2}{\hbar c}[\rho,\ \boldsymbol{R}] \tag{29}$$

and we obtain, refering to (24),

$$\boldsymbol{L}=\frac{1}{2}\frac{g^2}{c\hbar}\frac{c\varkappa}{\omega}\left\{1-\left[1-\left(\frac{\omega}{c\varkappa}\right)^2\right]^{3/2}\right\}\frac{1}{\omega}[\rho,\ \dot{\rho}]=\frac{1}{2\cos\theta}\frac{1}{\omega}[\rho,\ \dot{\rho}], \tag{30a}$$

or in components

$$L_3=\frac{1}{2}\frac{g^2}{\hbar c}\sin^2\theta\cdot\frac{c\varkappa}{\omega}\left\{1-\left[1-\left(\frac{\omega}{c\varkappa}\right)^2\right]^{3/2}\right\},$$

$$L=\frac{1}{2}(L_1+iL_2)=-\frac{1}{4}\frac{g^2}{\hbar c}\sin\theta\cos\theta\, e^{i\omega t}\cdot\frac{c\varkappa}{\omega}\left\{1-\left[1-\left(\frac{\omega}{c\varkappa}\right)^2\right]^{3/2}\right\} \tag{30b}$$

This result can also be deduced, treating the equation (28b) in the above mentioned rotating system and noting that $\rho=[\omega, \rho]$, $\dot{\boldsymbol{L}}=[\omega, \boldsymbol{L}]$.

In this way the charge vector of the excited self-field has been found. Now if the total charge is given to be $(m+\frac{1}{2})e$, then we have by means of (27) and (29a)

$$m=L_3+\frac{1}{2}\rho_3=\frac{1}{2}\frac{\sin^2\theta}{\cos\theta}+\frac{1}{2}\cos\theta \quad \text{or} \quad \cos\theta=\frac{1}{2m}. \tag{31}$$

In the quantum theory, m must be quantised as half-odd-integer, positive or negative. If we introduce this charge quantum number into our classical theory in a sense of correspondence principle, we may say that there occurs exact direction quantisation of the charge vector in the charge space, independently of strength of coupling. If however the coupling constant $g^2/\hbar c$ is considerably small as compared with 1, as experimental evidence shows, then no excited state exists even for the most favorable cases: $m=\pm\frac{1}{2}$ (first excited proton and neutron states), as mentioned already, though it seems yet questionable whether our classical theory still valid in the cases of such weak coupling.

On the contrary, if coupling is strong so that $g^2/\hbar c>1$, we have then the excited states even for $|m|>\frac{1}{2}$. In oder to get the change of the self-energy of the nucleon in such a state due to mesic self-field, we calculate the energy of the meson field, $\overline{H}'$, given by (14), (15) and (16) making use of (26). The result is

$$\overline{H}'=-\frac{1}{4}g^2x\left\{\cos^2\theta+\left[1+2\left(\frac{\omega}{cx}\right)^2\right]\sqrt{1-\left(\frac{\omega}{cx}\right)^2\sin^2\theta}\right\} \tag{32a}$$

which is alternatively written as, consulting (24a),

$$\overline{H}'=-\frac{1}{4}g^2x+\frac{3}{4}g^2x\left\{1-\sqrt{1-\left(\frac{\omega}{cx}\right)^2}\right\}\sin^2\theta-\frac{1}{2}\hbar\omega\frac{\sin^2\theta}{\cos\theta} \tag{32b}$$

The first term is the contribution from the neutral meson field and is the same as in the ground state. The charged meson field gives thus rise to the remaining two terms, of which the latter can be also deduced as follows: The total energy W must of course conserve: $W=E'+\overline{H}'=0$, and therefore we obtain from (18)

$$\overline{H}'=-\frac{g^2}{c}(\rho,\ \boldsymbol{R})\,. \tag{33}$$

On the other hand, since $\boldsymbol{L}$ rotates with the angular velocity $\boldsymbol{\omega}$ rigidly in the charge space, a contribution to the field energy is expected to amount to $C\cdot\frac{1}{2}(\boldsymbol{L},\ \omega)$, in analogy with the kinetic energy of a rotating rigid body. Remembering (29) and the relation $\rho=[\omega,\ \rho]$, comparison of the time rate of its change with (33) determines at once the constant C and yields $C=-2\hbar$, thus giving $-\hbar(\boldsymbol{L},\ \dot{\omega})$. This is nothing but the third term of (32b), only if we refer to (30a) and (24a). It is always negative in virtue of (24a), and is interpreted as a part of the interaction energy of the nucleon with the excited charged self-field. The last remaining second term of (32b) contains then in itself the proper energy of the charged field.

§ 5. Proton-isobars.

With the energy of the mesic self-field thus obtained, we now discuss whether proton-isobar can stably exist or not. Its excess mass over proton or neutron, i. e., the ground state of the nucleon, is from (31) and (32b),

$$\varDelta W_m=\mu\frac{1}{2}\,\frac{4m^2-1}{2|m|}\,x\left\{\frac{3}{2}\,\frac{1-\sqrt{1-x^2}}{1-\sqrt{1-x^{2/3}}}-1\right\},\quad (x\equiv|\omega_m|/c\varkappa)\,, \tag{34}$$

where μ is the meson mass.

The first excited states: $m=\pm\frac{1}{2}$ have therefore the same mass as for normal proton and neutron ($\varDelta W_{\pm1/2}=0$). In general it is observed that if we put $m=\pm(2n+1)/2$, where $n=0,\ 1,\ 2,\ldots$, proton-isobar with charge $+(n+1)e$ has necessarily the same mass as one with $-ne$. Now, in order that such a proton-isobar exist, stably against the emission of a free meson, the following condition must be to hold;

$$\varDelta W_m<\mu \quad \text{or} \quad 3\,\frac{\varkappa(n+1)}{2n+1}\,x\left\{\frac{1-\sqrt{1-x^2}}{1-\sqrt{1-x^{2/3}}}-\frac{2}{3}\right\}<1\,. \tag{35}$$

The function in the bracket is a monotone increasing function of x and has a zero point at $x=\sqrt{\frac{\sqrt{3}}{2}}$. (Cf. Fig. 1.) Its maximum value attained at $x=1$ amounts to 1/3 and therefore the condition (35) reduces to

$$\frac{n(n+1)}{2n+1} \leqq 1 \quad \text{or} \quad n \leqq \frac{1+\sqrt{5}}{2}$$

Thus besides the first excited states ($n=0$), only protons with singly negative and doubly positive charges are allowed to exist. As was already seen, the existence of them claims the coupling constant to be $g^2/\hbar c \geqq 3 \cdot 0{,}9785$. On the other hand, since these isobars must not, as a matter of course, be lighter than the normal proton and neutron, or $\Delta W_{\pm 3/2} \geqq 0$, it must be $x \geqq \sqrt{\frac{\sqrt{3}}{2}}$ and this affords another condition for $g^2/\hbar c$. It comes out the same condition as the given in above, for, as is stated in §3, $x=\sqrt{\frac{\sqrt{3}}{2}}$ corresponds to the critical frequency for $g^2/\hbar c = 3 \cdot 0{,}9785$, from which the equation (24a) just begins to have solution. Thus, from the above value of coupling consistant on to 3, there exist certainly one solution, which fulfils the requirement. But there also appears another possible one, which remains to exist even for larger values of coupling constant and therefore leads to isobar-catastrophe, say, in a sense that isobars with higher charges make themselves more and more stable. Even for $g^2/\hbar c = 3 \cdot 0{,}9785$, all isobars have the same mass as the normal proton. However the experimental evidence shows that $g^2/\hbar c \sim 0.1$ and thus the catastrophe does not occur, and negative proton might probably not exist.

Finally we briefly mention as regards the so-called cut-off method. In his theory, Tomonaga, l. c., obtained a characteristic equation for ω, in which an integral containing the cut-off factor $F(k)$ appeared. If we evaluate this integral and neglect the quantities of higher orders, we arrive at

$$\frac{\omega}{c\varkappa} = \frac{g^2}{\hbar c} \cos\theta \left\{ -\frac{2}{\pi} \left(\frac{\omega}{c\varkappa}\right)^2 \frac{K}{\varkappa} + 1 - \left[1 - \left(\frac{\omega}{c\varkappa}\right)^2\right]^{3/2} \right\},$$

where K is the cut-off momentum devided by $\hbar c$. The introduction of the cut-off factor is equivalent to giving the nucleon a finite radius of order K^{-1}, at least in a non-relativistic approximation of the theory. In fact, if we do so, thus making the singular terms in the expressions of the flux from the surface of the nucleon so as to remain finite, then $\boldsymbol{R}$ given by (12) gains an additive term: $-\frac{K}{2c\varkappa^2}\ddot{\rho}$. (The another additive term which is already finite vanishes itself for our case considered here.) For our excited state (22), this turns out to be $-\left(\frac{\omega}{c\varkappa}\right)^2 \frac{K}{\varkappa}$, which is just the first term in the bracket in the above equation. (The similar correspondence is evidently observed also for the charge vector of the meson field, L_3.) Now, if we take K large compared with $\varkappa$, as is usually assumed, the above inertia term does mostly predominate and makes the other terms almost impotent. Thus the precessional motion of the charge vector becomes always slow; proton-isobars also have small exess masses and fall each other very densely in mass scale.

In conclusion, I express my gratitude to Mr. S. Ozaki for his kind discussions. The research was aided by the Scientific Research Expenditure of the Department of Education.

Physical Institute,
Tôhoku University.

References

(1) W. Heisenberg, ZS. Physik, **113** (1939), 61.
G. Wentzel, Helv. Phys. Acta, **13** (1940), 269.
S. Tomonaga, Sci. Pap. I.P.C.R., **39** (1941), 247.
J. R. Oppenheimer & J. Schwinger, Phys. Rev., **60** (1941), 150.
T. Miyazima & S. Tomonaga, Sci. Pap. I.P.C.R., **40** (1942), 21; ibid., **40** (1943), 274.
S. Tomonaga, Bull. I.P.C.R., **22** (1943), 711, 781.
S. Kanezawa, H. Jen & S. Tomonaga, Bull, I.P.C.R., **23** (1944), 81.
S. Ozaki, Prog. Theor. Phys. **1** (1946), 56.
M. Born, Proc. Roy. Soc. London, **A 143** (1934), 410.
(2) M. Born & L. Infeld, ibid., **A 144** (1934), 425.
F. Bopp, Ann. Physik, **38** (1940), 345.
Cf also E. Kanai and S. Takagi, Prog. Theor. Phys., **1** (1946), 43.
(3) P. A. M. Dirac, Proc. Roy. Soc. London, **A 167** (1938), 148.
M. H. L. Pryce, ibid., **A 168** (1938), 389.
(4) H. J. Bhabha, Proc. Roy. Soc. London, **A 172** (1939), 384; ibid., **A 178** (1941), 314.
H. J. Bhabha & H. C. Corben, ibid., **A 178** (1941), 273.
(5) M. Fierz, Helv. Phys. Acta, **14** (1941), 257.
(6) W. Heitler, Proc. Cambridge Phil. Soc., **37** (1941), 291.
A. H. Wilson, ibid., **37** (1941), 301.
Cf. also S. Tomonaga, Sci. Pap. I.P.C.R., **40** (1942), 73; G. Araki, ibid., **40** (1943), 311.

Note added in Proof: After sending the manuscript, the author had occasion to see the Pauli's paper (Phys. Rev., **64**, 332, 1943) and also the Dirac's paper refered there (Ann. de l'Inst. Poincare, **9**, 13, 1939). In the former, the problem of spin proton-isobars was discussed for the case of neutral pseudoscalar meson, applying the λ-limiting process of Wentzel and Dirac which is regarded as a mathematical formulation of the Dirac's subtraction method.

Progress of Theoretical Physics Vol. II, No. 4, Nov.~Dec., 1947.

On the Electromagnetic Properties of Nucleons, III.*

By Eizo KANAI and Minoru KOBAYASI

(Received May 5, 1947)

§ I. Introduction.

In the previous two papers,*) we have discussed the electromagnetic properties of nucleons from the different standpoints. In the first paper, we have investigated these properties by calculating the probabilities of transitions, which are caused by the radiation fields, between two spin states of nucleons, and obtained the finite results which should be interpreted as the virtual electric moments of nucleons. In the next paper, we have directly calculated the additive energy of the nucleon placed in the homogeneous magnetic field by taking recourse to the subtraction prescription which is analogous to the theory of positrons. By this calculation also, we have not obtained the so-called anomalous magnetic moments of nucleons, but their finite magnetic polarizabilities.

However, that the nucleons have the anomalous magnetic moments is the experimentally well established fact. We shall, therefore, closely investigate this problem once more in this paper by calculating the transition probabilities of another process which are directly examined by the experimental method. It must be remembered here, that there are two distinct methods of measuring the magnetic moments of elementary particles, one of which was used by Stern and Gerlach, and the other is that of Rabi and his coworkers. By the later method one measures the energy difference of two spin states of particles placed in the homogeneous magnetic field. In order to interprete the result thus obtained we must estimate the energy difference of two staitionary states. But, as was shown by many authors, these calculations lead to the diverging results in the present formalism of the quantum field theory, and it is necessary to introduce some new procedures ad hoc to avoid this divergence as was tried in II. Nevertheless, the result obtained in this way seems to be very ambiguous, because of the unsatisfactory nature of the present theory especially in the computation of the energy values of the stationary states.

---

* The contents of this paper were read at the symposium of the Physico-mathematical society of Japan held at Kyoto University, on March 2, 1946.

On the contrary, the former experimental method which measures the deviation of the path of the elementary particle in the inhomogeneous magnetic fields is free from ambiguity in its theoretical interpretation. In fact, the quantum electrodynamics has succeeded in the various transitional problems so long as we are concerned in the first non-vanishing approximations. This circumstance suggests us what is the useful portion of the present formalism of the field theory.

Hence, we perform a "Gedankenexperimenz" here. For simplicity, let us suppose a neutron with momentum $\overrightarrow{P_1}$ and energy $E_1(>0)$ set in the inhomogeneous magnetic fieid $\overrightarrow{H}$. As the neutron has no charge nor Dirac's magnetic moment, the transition from this initial state to the state of momentum $\overrightarrow{P_2}$ and energy $E_2(>0)$, if possible, must entirely due to its additional electromagnetic properties.

Next, we try the theoretical explanation of above experiment. As the mechanism of the deviation of the neutron path is to be determined by the virtual dissociation of the neutron state into the state where a proton and mesons exist, we have only to consider the interaction between the nucleon plus meson field and the external electromagnetic field. Here, we take the following assumptions for simplisity.

(1) the Schrödinger approximation with spin for the neutron,

(2) the pseudo-scalar formalism for the meson field.

(1) is justified, for the speed of the neutron is very slow in the thermal region. Further, if we adopt the relativistic equation for the neutron, we must use the positron theory in the same time, but it is attended with danger of introducing another difficulty. Moreover, in Stern-Gerlach's experiment the meaning of the spin is more clear in the form of Pauli's spin than in Dirac's form. The assumption (2) is used because of our expectation of the spin-dependent magnetic moment.

In the following calculations, we must content ourselves with the first non-vanishing result by the reason above discussed. Further, in this paper, we shall perform the analogous consideration for the nucleon set in the inhomogeneous electric field and deduce the electrical properties of the nucleon as well.

§ II. Calculations of the materix-elements.

The system composed of a nucleon, meson field and electromagnetic field is described by the following Hamiltonian(2);

$$H=H^N+H^M+H^{NM}+H^{NE}+H^{ME}+H^{NME}+\text{const.}$$

where

$$H^N=\frac{1}{2m}\vec{P}^2$$

$$H^M=\int\left(4\pi\varkappa^2c^2\Pi^*\Pi+\frac{1}{4\pi}\psi^*\psi+\frac{1}{4\pi\varkappa^2}\nabla\psi^*\cdot\nabla\psi\right)d\tau$$

$$H^{NM}=-\frac{g}{\varkappa^2}\left(O^*\vec{\sigma}\cdot\nabla\psi^*+O\vec{\sigma}\cdot\nabla\psi\right)\equiv H^g$$

$$H^{NE}=-\tau_P\left\{\frac{e}{mc}(\vec{A}\cdot\vec{P})+\frac{e\hbar}{2mc}(\vec{\sigma}\cdot\vec{H})-eA_0\right\}\equiv H^{e'}$$

$$H^{ME}=\int\left\{\frac{ie}{\hbar}A_0(\Pi^*\psi^*-\Pi\psi)+\frac{ie}{4\pi\varkappa^2\hbar c}(\vec{A}\cdot\psi^*\nabla\psi-\vec{A}\cdot\psi\nabla\psi^*)\right.$$
$$\left.+\frac{e^2}{4\pi\varkappa^2\hbar^2c^2}A^2\psi^*\psi\right\}d\tau\equiv H^e$$

$$H^{NME}=\frac{ige}{\varkappa^2\hbar c}\left(O\vec{\sigma}\cdot\vec{A}\psi-O^*\vec{\sigma}\cdot\vec{A}\psi^*\right)\equiv H^{ge},$$

and ψ and Π are canonical variables of the meson field, $\vec{\sigma}$, O and O^* are the usual spin and isotopic spin ,operators of the nucleon and τ_P is the charge operator whose eigenvalue is 1 for the proton state and 0 for the neutron state. A_0 and $\vec{A}$ are the scalar and vector potentials of the electromagnetic field respectively. m and $\mu=\hbar/c\varkappa$ are the masses of the nucleon and the meson respectively. g and e are the coupling parameters as seen above.

Now we consider as the wave function for the state where the neutron has the momentum $\vec{P_1}$ and energy E_1, associated by the proper meson field,

$$\Phi_1(g)=\Phi_i+\sum{}'\frac{H^g_{ni}}{E_i-E_n}\Phi_n+\left(\sum_{n.m}{}''\frac{H^g_{nm}H^g_{mi}}{(E_i-E_m)(E_i-E_n)}\Phi_n\right.$$
$$\left.-\frac{1}{2}\sum_n{}'\frac{H^g_{in}H^g_{ni}}{(E_i-E_n)^2}\Phi_i\right)+O(g^3)$$

where Φ_i is the neutron state having no meson field around it, while Φ_n is the nucleon state with some mesons virtually emitted, which are both the eigenstates of H^N+H^M. In the same way, we find the wave function for the final state where the neutron has the momentum $\overrightarrow{P_2}$ and energy E_2 as

$$\Phi_2(g)=\Phi_f+\sum_n{}'\frac{H^g_{nf}}{E_f-E_n}\Phi_n+\left(\sum_{n.m}{}''\frac{H^g_{mn}H^g_{mf}}{(E_f-E_m)(E_f-E_n)}\Phi_n\right.$$
$$\left.-\tfrac{1}{2}\sum{}'\frac{H^g_{fn}H^g_{nf}}{(E_f-E_n)^2}\Phi_f\right)+O(g^3).$$

So the transitional matrix element becomes as follows;

$$(\Phi_2(g)/H^e+H^{e'}+H^{ge}/\Phi_1(g))$$

which contains the terms proportional to even powers of g such as g^2, g^4 etc. Applying the prescription that we should leave only the first non-vanishing approximation in the perturbation method, we have

$$H_{fi}=\sum_n{}'\frac{H^{ge}_{fn}H^g_{ni}}{E_i-E_n}+\sum_n{}'\frac{H^g_{fn}H^{ge}_{ni}}{E_f-E_n} \tag{a}$$

$$+\sum_{n.m}{}''\frac{H^e_{fn}H^g_{nm}H^g_{mi}}{(E_i-E_m)(E_i-E_n)}+\sum_{m.n}{}''\frac{H^g_{fm}H^g_{mn}H^e_{ni}}{(E_f-E_m)(E_f-E_n)}+\sum_{n\,m}{}''\frac{H^g_{fn}H^e_{nm}H^g_{mi}}{(E_f-E_n)(E_i-E_m)} \tag{b}$$

$$+\sum_{n.m}{}''\frac{H^g_{fn}H^{e'}_{nm}H^g_{mi}}{(E_f-E_n)(E_i-E_m)}. \tag{c}$$

(i) Case of the inhomogeneous magnetic field:

$$\overrightarrow{H}=\operatorname{rot}\overrightarrow{A},\qquad \operatorname{div}\overrightarrow{A}=0,\qquad A_0=0.$$

In this case we obtain

$$(\mathrm{a}):\ H^a_{fi}=-\frac{2}{3\pi}\frac{\hbar eg^2}{\mu^2c^3}\eta\int\varphi^*(\overrightarrow{P_2})\left[\int_0^\infty\frac{k^4dk}{k_0(k_0+\eta k^2)^2}\right.$$
$$\left.\times\left\{\overrightarrow{A}\cdot(\overrightarrow{k_1}+\overrightarrow{k_2})+(\overrightarrow{\sigma}\cdot\overrightarrow{H})\right\}\right]\varphi(\overrightarrow{P_1})d\overrightarrow{x}$$

$$(\mathrm{b}):\ H^b_{fi}=\frac{1}{3\pi}\frac{\hbar eg^2}{\mu^2c^3}\eta\int\varphi^*(\overrightarrow{P_2})\left[\int_0^\infty\frac{(3k_0+\eta k^2)k^8}{k_0^3(k_0+\eta k^2)^3}dk\right.$$
$$\left.\times\left\{\overrightarrow{A}\cdot(\overrightarrow{k_1}+\overrightarrow{k_2})\right\}\right]\varphi(\overrightarrow{P_1})d\overrightarrow{x}$$

$$+\frac{1}{3\pi}\frac{\hbar e g^2}{\mu^2 c^3}\int \varphi^*(\vec{P_2})\left[\int_0^\infty \frac{(2k_0+\eta k^2)k^4}{k_0^3(k_0+\eta k^2)^2}dk(\vec{\sigma}\cdot\vec{H})\right]\varphi(\vec{P_1})d\vec{x}$$

$$(\mathrm{c}):\ H_{fi}^{c}=-\frac{1}{3\pi}\frac{\hbar e g^2}{\mu^2 c^3}\eta\int \varphi^*(\vec{P_2})\left[\int_0^\infty \frac{(3k_0-\eta k^2)k^4}{k_0(k_0+\eta k^2)^3}dk \times\left\{\vec{A}\cdot(\vec{k_1}+\vec{k_2})\right\}\right]\varphi(\vec{P_1})d\vec{x}$$

$$+\frac{1}{3\pi}\frac{\hbar e g^2}{\mu^2 c^3}\eta\int \varphi^*(\vec{P_2})\left[\int_0^\infty \frac{k^4 dk}{k_0(k_0+\eta k^2)^2}(\vec{\sigma}\cdot\vec{H})\right]\varphi(\vec{P_1})d\vec{x}$$

where we put $\vec{P_1}=\hbar\vec{k_1}$, $\vec{P_2}=\hbar\vec{k_2}$, $k_0=\sqrt{k^2+\varkappa^2}$, $\eta=\frac{\hbar}{2mc}$ and used the approximation $\frac{1}{2m}P_1^2,\ \frac{1}{2m}P_2^2\ll\mu c^2$ which is justified by the assumption (1) of § 1. Further $\varphi(\vec{P})$ is the function of the neutron belonging to the momentum $\vec{P}$.

As is easily seen, H_{fi} has two kinds of terms, one of which is dependent upon the spin of the neutron while the other is not. We shall investigate each term separately.

(1) spin-dependent term:

$$H_{fi}=-\int \varphi^*(\vec{P_2}).(M\vec{\sigma}\cdot\vec{H})\varphi(\vec{P_1})d\vec{x}$$

where

$$M=\frac{1}{3\pi}\left(\frac{\hbar e g^2}{\mu^2 c^3}\right)\int_0^\infty dk\left[\frac{2\eta k^4}{k_0(k_0+\eta k^2)^2}-\frac{(2k_0+\eta k^2)k^4}{k_0^3(k_0+\eta k^2)^2}-\frac{\eta k^4}{k_0(k_0+\eta k^2)^2}\right]$$

$$=-\frac{1}{3\pi}\left(\frac{\hbar e g^2}{\mu^2 c^3}\right)\int_0^\infty \frac{(2k_0-\eta k^2)k^4}{k_0(k_0+\eta k^2)^2}dk.$$

(2) spin-independent term:

$$H_{fi}=-\int \varphi^*(\vec{P_2})\left\{\frac{\varepsilon}{mc}(\vec{P}\cdot\vec{A})\right\}\varphi(\vec{P_1})d\vec{x}$$

where

$$\varepsilon=\frac{2}{3\pi}\eta\left(\frac{m c g^2}{\mu^2 c^3}\right)\int_0^\infty dk\left[\frac{2k^4}{k_0(k_0+\eta k^2)^2}-\frac{(3k_0+\eta k^2)k^6}{k_0^3(k_0+\eta k^2)^3}+\frac{(3k_0-\eta k^2)k^4}{k_0(k_0+\eta k^2)^3}\right]$$

$$=\frac{1}{3\pi}\left(\frac{\hbar e g^2}{\mu^2 c^3}\right)\int_0^\infty \frac{k^4 dk}{k_0^3(k_0+\eta k^2)^3}(2k_0^3+\eta\varkappa^2 k^2+3\varkappa^2 k_0).$$

Immediately, (1) and (2) enable us to interpret the meaning of $\overrightarrow{M\sigma}$ or $\frac{\varepsilon}{m}\overrightarrow{P}$. Namely, we can interpret $\overrightarrow{M\sigma}$ as the anomalous magnetic moment of the neutron and $\frac{\varepsilon}{m}\overrightarrow{P}$ as its additional current.

(ii) Case of the inhomogeneous electric field.

$$\overrightarrow{E}=-\operatorname{grad} A_0, \qquad \overrightarrow{A}=0.$$

From the analogous computation as the above one, we obtain.

(1) spin-dependent term:

$$H_{fi}=-\int\varphi^*(\overrightarrow{P_2})(\overrightarrow{N\sigma}\cdot\overrightarrow{E})\varphi(\overrightarrow{P_1})\,d\overrightarrow{x}$$

where

$$N=0.$$

(2) spin-independent term:

$$H_{fi}=\int\varphi^*(\overrightarrow{P_2})(\mathrm{e}A_0)\varphi(\overrightarrow{P_1})\,d\overrightarrow{x}$$

where

$$\mathrm{e}=0.$$

So it can be said that the neutron has no additional charge and electric moment.

§ III. Determination of the coupling constant.

From the previous calculation, we find for the neutron to have the following anomalous magnetic moment

$$M\sim-\frac{2}{3\pi}\frac{\hbar e g^2}{\mu^2 c^3}\int_0^\infty\frac{k^4\,dk}{k_0^2(k_0+\eta k^2)^2}\doteqdot-\frac{4}{3\pi}\frac{e g^2}{\mu c^2}\left(\frac{m}{\mu}\right)$$

or, measuring in the unit of nuclear magneton $\overrightarrow{\mu_0}$,

$$\overrightarrow{M\sigma}\approx-\frac{8}{3\pi}\left(\frac{g^2}{\hbar c}\right)\left(\frac{m}{\mu}\right)^2\overrightarrow{\mu_0}.$$

By the comparison with the experimental data, for example, given by Bloch and Alvarez(3) whose results show that $\overrightarrow{M\sigma} = -1.94\overrightarrow{\mu_0}$, we can determine the coupling constant g or the mesonic mass μ.

First we shall take $m = 10\mu$, then we get

$$\frac{g^2}{\hbar c} \approx \frac{1}{44} \quad \text{or} \quad g \approx 1.8e$$

This value of g is considerably small when compared with the previous value required by the nuclear potential. But, we must remember that the nuclear potential has to be complemented by the cut-off procedure, while our conclusion is free from such a procedure.

Next, we shall take $g = e$ for the simplicity of the coupling scheme, and get

$$\mu \approx \frac{1}{18} m \doteqdot 100 m_e$$

where m_e is the mass of the electron. We cannot exclude this value of μ from other data in the present stage of the nuclear physics. At any rate, the small value of g — which is consistent with the idea of the weak coupling (perturbational treatment) — is the characteristic feature of the present calculation.

Finally, we shall discuss about the additional current $\frac{\varepsilon}{m}\overrightarrow{P}$ briefly. Though ε is not the charge, it has the dimension of the charge. So we can speak of ε in the place of $\frac{\varepsilon}{m}\overrightarrow{P}$ and compare it to the elementary charge e, i. e.

$$\varepsilon \simeq \frac{2}{3\pi} \frac{\hbar e g^2}{\mu^2 c^3} \int_0^\infty \frac{k^4 dk}{(k_0 + \eta k^2)^3} \doteqdot \frac{4}{3\pi} \frac{e g^2}{\hbar c} \left(\frac{m}{\mu}\right)^2$$

Putting $\frac{g^2}{\hbar c} = \frac{1}{44}$, $m = 10\mu$, we obtain

$$\varepsilon \approx 0.09e.$$

Therefore we cannot expect the remarkable effect of ε for the thermal flow of the neutron beam within the magnetic field, that is consistent with real experiments. If we had assumed the alternative values $g = e$ and $\mu = \frac{1}{18} m$, we would obtan $\varepsilon \approx e$, which would not also show the appreciable effect in the slow neutron experiments.

In conclusion, we should like to express our deep gratitude to Mr. S. Kokaji and Mr. S. Takagi for their kind help with this calculation.

Department of Physics, Kyoto University.

References:

(1) Kobayasi and Kanai: Prog. Theor. Phys. **1** (1946), 21.
Kobayasi, Kanai and Kokaji: Prog. Theor. Phys. **2** (1947), 151. This paper shall be cited as II.

(2) Tanikawa and Yukawa: Proc. Phys-Math. Soc. of Japan **23** (1941), 445.

(3) Alvarez and Bloch: Phys. Rev. **57** (1940), 111.

Progress of Theoretical Physics Vol. II, No. 4, Nov.~Dec., 1947.

Conflict between Matter and Field.*

—An Analysis of the Difficulties of the Theory of Elementary Particles—

Mituo TAKETANI.

(Received April 12, 1947)

The present status of theoretical physics is confronted with difficulties of extremely ambiguous nature. These difficulties can be glossed over in various ways, but no one believes that a definite solution has been attained. The reason for this is that on one hand present theoretical physics itself has logical difficulties, while on the other, there is no decisive experiment whereby to determine the theory uniquely. In other words, the present difficulties manifest themselves in that the theory itself involves certain diverging results, as well as deviating a little from corresponding experiments; and they are as yet not adequately sorted out. But it is generally accepted that these difficulties cannot be overcome by anything in the way of a trivial modification. In short, the state of affairs is just like that on the eve of the advent of quantum mechanics. For in that case too, the discrepancies with experiment were, quantitatively speaking not so great. But unlike those on the eve of quantum mechanics, the present difficulties are not yet sorted out into a radical form such as the incompatibility and inconsistency between fundamental concepts, so that there seems to be some uncertainty as to the point at which to commence attack of the problem. We must set these in order, run them down into an inconsistency between basic concepts, and ascertain the shape of the inconsistency, thus paving the way to a new theory. Recent developments (up to the early stages of the Pacific War) have furnished much material for this purpose. The present inconsistencies and difficulties are believed to touch the internal structure of elementary particles themselves. I believe that the clue to an analysis at this juncture is provided by identifying this fundamental inconsistency between concepts with the conflict between matter and field.* The two fundamental concepts of matter and field are those most essentially standing against each other at present, embodying the most essential problem of contemporary theory—just as the two incompatible concepts of wave and particle gave birth to quantum mechanics. Much beating about the bush having been made in connection with this problem. Now it becomes the central problem and its definite solution is an urgent matter of the day.

---

* This is a synopsis of what I made public at the "Nuclear Physics Colloquium" in May 1943, and also what was discussed with Prof. S. Sakata and others at the Physical Department of the Nagoya University in November 1945. Although it is by no means perfect yet, I have decided to publish it, hoping it might furnish some information or other, and also that it might lead to another series of discussions. A concrete analysis, forming the latter half in the MS, has been omitted, be ingreserved for another opportunity. I extend my sincere thanks to Prof. Tomonaga and his coworkers as well as to Prof. Sakata and others for having discussed the matter over with me, and also to Mr. M. Tatsuoka who has helped to prepare present paper in this form.

It is a well-known fact that, in the formulation of electron theory, the energy of the field becomes infinitely large unless the electron in classical theory is taken to possess a finite radius, so Lorentz considered a model in which the charge was distributed over a small sphere. As a result, however, forces interact between each charge density, so in order that the electron be kept rigid notwithstanding, a sort of cohesive force must be introduced. This is no longer an electro-magnetic force.(1) Therefore, the necessity arises of considering a point model of the electron. Later on, in quantum mechanics the superposition of material waves was considered in order to denote the existence in space of an electron, in which case no "size" could be attributed to it. So here again was a point-model required. In a point electron, firstly, Lorentz's equation of motion cannot be applied at the place where the charge exists, as the field strength becomes infinitely large there. Secondly, as mentioned above, we obtain an infinitely large electro-magnetic energy at the point of charge.

These difficulties are not confined to classical theory alone, but in quantum theory too, in quantum electro-dynamics, they are present in the fact that a point electron possesses an infinitely large interaction with itself. The above difficulties in classical electron theory and quantum electrodynamics were, however, rather merely theoretical, and had the character of an "academic problem" as it were. For these difficulties offer no hindrance in explaining theoretically all the experimental facts we obtain concerning an electron, and they had significance only as a problem in themselves, independent of experimental facts. The only experimental fact in this connection is the mass m of an electron.

---

* This is, of course, a conjecture which is to be tested in the process of actual analysis by its success or failure, and gradually the correct key will be found.

This much is, however, certain: that this suffices to summarize the many attempts of various physicists up to now, to clarify their results, and to furnish some clue to future analysis.

** Matter, here is no philosophical concept. That is, it does not imply materialism, but is a common term among physicists, meaning particles which act as sources of fields, for example electrons, protons, neutrons, charged mesons and so on.

But there have been much meandering concerning this question. That is to say, until mesons were discovered within cosmic-rays, the large penetrating power of the latter believed the fact that, according to Dirac's relativistic equation for an electron, the energy loss should be great when high-energy electrons penetrate through matter. Also, the experimental fact that there was a phenomenon in cosmic-rays called shower, in which many particles are produced explosively, it seemed, in a single process.* could not be explained. Therefore, these difficulties came to be ascribed to the same cause as the above-mentioned difficulties in electron theory and quantum electro-dynamics. To these were further associated the difficulty of the nuclear electron, which was considered in relation with β-decay. However, when the meson was discovered and the large penetrating power of cosmic-rays attributed to it. and the theory of cascade shower was established at about the same time (1937) doing away with the necessity of considering showers as single processes explosively producing many negative and positive electrons, thus making Dirac's equation acceptable even for extremely high energies, once more did the fundamental difficulty of electron theory and quantum electrodynamics come to lose its realistic foothold.

But as meson theory made further progress, the quantum electrodynamical difficulty made its second appearance—in a new light: the field theory was developed in a manner similar to electron theory, and, evidently in compliance with the correspondence principle, the meson field was quantized in the same way as in quantum electrodynamics. Thus, it was natural that the same difficulty presented itself here as in the quantum electrodynamical case.

In this case, however, the difficulty turned out to be more realistic and abundant in experimental evidence—as I frequently emphasized at the time we** were performing the quantization of the meson field (1937–8). One of the experimental evidences is the anomalous magnetic moment of heavy particles (protons and neutrons), which is obtained, in accordance with perturbation theory, by assuming intermediate states and considering mesons to dissociate around heavy particles and interact with the magnetic field. This involves a calculation similar to that for self-energies, so that the result diverged just as the self-energies did. Thus, unlike in the case of the electron, the difficulty in the case of mesons had a close connection with experimental fact. It was for this reason that I stated in those days :(2) " This time for certain, it will be impossible to develop meson theory leaving this difficulty as it is ".

---

* It was later ascertained experimentally that this is not a single process.

** Drs. Yukawa, Sakata and I.

To cope with this difficulty, the method of " cut-off ", in which the momentum integration of the intermediate state is cut off at a certain value in order to prevent its diverging, has been employed (It is the same as inserting a suitable converging factor).

Later on as the meson theory made further development, it was found that when vector and pseudoscalar mesons were scattered by heavy particles, and also when charged vector mesons interacted with an electromagnetic field, the scattering cross-section became larger as the energy increased, which contradicts experiment. Further, while the interaction due to the Coulomb field about an electron is $-\frac{e^2}{r}$ (where e is the charge and r the distance), an interaction term $-g\frac{2\epsilon^{-\kappa}}{r^3}$ (where g is the constant of interaction between a meson and a heavy particle, and $x=\frac{m_u c}{\hbar}$ [m_u : meson mass, c : light velocity, $\hbar$: planck's constant $\div 2\pi$]) presents itself in the meson field surrounding a heavy particle, and such a singularity brings about a difficulty in the solution of the equation of motion. Hence the method of " cut-off ", in which the interaction is cut off for momentam above a certain value, becomes necessary in various cases. And since momentum implies wave length, the cutting off at a certain momentum value involves the necessity of introducing some sort of an universal length, as Heisenberg emphasize (3)

Later, the problem developed in a rather different direction, and assumed a more concrete form. In the case of strong interactions like that of the meson field, it is natural that the applicability of perturbation theory be doubted ; and even if perturbation theory can be used, it is naturally to be expected that a damping term must be introduced, as in the case of deducing a dispersion formula. And I have thought that some sort of a dispersion formula would be given by the correct theory. A similar line of thought was taken by Landau.(4)

In this way, various trials were made to depart from the perturbation method of radiation fields. For instance, a classical solution of the equations was carried out, and it was made clear that certain important parts of the difficulty had their origin in an inappropriate use of perturbation theory.

This was performed by Heisenberg,(5) Bhabha,(6) Wentzel,(7) and Tomonaga.(5)

Also it was shown by Heither(9) and Wilson(10) that the divergence could be removed in the case of high energy scattering if radiation damping was taken into account in perturbation calculations of radiation theory.

Thus it was found that a part of the above-mentioned difficulty arose because radiation damping had not been taken into account, and the state of affairs assumed a rather different aspect. Many previous calculations now became meaningless. Perturbation theory could no longer be used without due precaution. For instance the method of Christy and Kusaka(11) which attempts to determine the meson spin from the interaction between mesons and an electro-magnetic field would give according to Wilson,(10) a value too much definite to be anticipated.

In the perturbation theory of radiation, the difficulty of field theory does not present itself in the first term. It is well known that this occurs in the second term. The afore-mentioned difficulty which was removed by considering radiation damping is one relating to this first term, and the second term which involves self-energies still remains divergent. Thus the introduction of radiation damping seemed to have brought about success in cornering the difficulty to a certain extent. But in meson theory, the interaction is strong, and much doubt is to be entertained in a perturbation method where phenomena involving a single meson alone come into the first term. So a solution which lays more stress upon phenomena involving many mesons at once comes to be sought for. This was what Heisenberg previously considered in connection with his universal length relating to the theory of bursts, and a brilliant solution was achieved by Tomonaga.(12) As frequently extorted by Tomonaga, it is not sufficient merely to introduce damping in the calculations, as done by Heither and Wilson. When the perturbation method becomes inavailable, this second and diverging term must, of necessity, be touched. Tomonaga used Fock's method and solved the equation for a meson in configuration space. Making calculations by this method for the case of mesons scattered by heavy particles,(13) it was found that self-energy played an important rôle in the case of scattering too. This takes the form, as anticipated, of a dispersion formula.*

In this formula, a diverging self-energy enters as resonance energy. Also the radiation damping enters as the width of resonance. This self-energy must be taken into consideration, as it becomes infinitely large. It is impossible in this calculation, to apply the subtraction method as in Dirac's theory. On the other hand, if it is made finite by the method of "cut-off", it acts as the energy deviation in the scattering process, and as the inertia of the proper field advocated by Heisenberg. This is directly connected with the scattering cross-section for low-energy mesons, and improves the agreement with experiment. In this way, divergency problems related to self-energy called for solution, coming to the foreground enbloc in an internally rearranged form. That is to say, it become a theme in hand to solve this problem of "field and matter".

Various standpoints have hitherto been taken concerning the problem of matter and field. The usual method adopted in quantum mechanics is to treat matter and field equivalently, consider their respective equations of motion, and introduce a term expressing their interaction. This has been done by Heisenberg and Pauli, and leads to the well-known divergence of self-energy. In this standpoint the interaction between two fields is considered, and there is no differentiating which is matter and which is field. Only when one associates the rest-mass of one of the fields with its self-energy, does one regard that field as matter, and the other, that causes self-energy from the interaction with the former, is regarded as field. This involves the divergence of self-energy, to cope with which a cutting-off or the artificial convergence factor is introduced, as done by Watagin(14)

---

* The scattering cross-section for charged vector mesons is

$$Q=\frac{\pi}{k_0^2}\,\frac{4(2\pi l)^4 k_0^6}{\left(K_0-l^2\int\frac{k^2}{K(K_0-K)}\,dk\right)^2+(2\pi l)^4 k_0^6}=\frac{\pi}{k_0^2}\,\frac{4(\Gamma/2)^2}{(K_0-K^{(0)})^2+(\Gamma/2)^2}$$

where k is the momentum $(k_1,\ k_2,\ k_3)$ k_0 the momentum of the incident meson, $\kappa=mc/\hbar$

$$K_0=\sqrt{k_0^2+\kappa^2},\quad l=\frac{1}{2\pi}\,\frac{g_1}{\sqrt{\hbar c}}=\frac{1}{\kappa}$$

This equation is a dispersion formula with resonance energy $K^{(0)}=l^2\int\frac{k^2}{K(K_0-K)}dk$ and $\Gamma/2=(2\pi l)^2 k_0^3$ its amplitude.

and Scherzer(15) in early stages. In either case the result is that something of the dimension of length is introduced in the interaction term, which leads us to anticipate a contact with the question of the structure of elementary particles. Of course it is evident that this length is not of such a purely substantialistic nature as Lorentz' electron radius, but it is also certain that it touches, in some sense or other, the question of structure.

Now, as the logical structure of quantum mechanics deals persistently with the existence probability of matter regarded as a point, the logical structure of observation is so built up that it is impossible to take up separately into question the "left or right half" of matter regarded as having some sort of magnitude or other. In other words, quantization leads to the existence probability of particles regarded as geometrical points. Unless this framework is altered, it would be impossible to deal with the structure of elementary particles. Of course "structure" in this case is not merely a spatial, but a spatio-temporal structure, and is intimately connected with the laws of relativity. The above mentioned standpoint of the interaction of two fields cannot deal with such a spatio-temporal structure. In particular the magnitude of spin (angular momentum of rotation) has a quantum number ½ in objects regarded mainly as matter.

This value ½ follows as the natural consequence of the group theory of spatial rotations, and though the operator representing angular momentum is incommutable with space co-ordinates when its quantum number is 1, it is commutable with space-time co-ordinates in this case. Although spin does not, of course, signify any directly perceptible rotation of a body with extension, the spin ½ is, in this sense, to be associated only with the existence probability of an elementary particle, and not with its spatio-temporal structure.

Markov,(16) in the theory of field quantization, postulated characteristic commutation-relations between the four-dimensional co-ordinates of the test-body employed to measure the field strength and the field quantities, introducing in them a length corresponding to the radius of the test-body, with some success over the divergence difficulty. Markov's theory shows that the mutual interaction type of treatment current in quantum mechanics is inapplicable to observations concerning the structure of elementary particles unless due modifications be made; and also that when such observations are in question, the space-time co-ordinates no longer merely specify the point of existence of matter, but possess organic inter-relations with field quantities, suggesting that the observation should be associated with the structure of elementary particles. But his theory does not penetrate into the question of spin, and also the mode of observation is not radically re-established but simply inherited from quantum mechanics.

Applying the method of Dirac, Fork and Podolsky's(17) "many-time theory", Wentzel(18) has obtained the difference between retarded and advanced fields by causing the test-body to approach the particle from a time-like direction, thus attaining some success in preventing the self-energy divergence. We learn, from papers which came to hand after the termination of the war, that Dirac has adopted this method in the form of the λ-limiting process, which is further developed by Pauli.(19) Using this, and at the same time introducing negative probability and pnotons with negative energy, Pauli has succeeded, at least formally, in removing all the divergence difficulties.

This is quite interesting from the consequential viewpoint, and especially noteworthy is the fact that certain divergences, which cannot be removed by either the λ-limiting process or the introduction of negative probability and energy alone, vanish on a simultaneous application of both these methods. However, there is a possibility that this might considerably affect hitherto satisfactory results too; and also, the comparison with experiment of the results cleared of divergencies is still incomplete. So nothing can be said, as yet, even of the results themselves. Furthermore, the structure of the theory is not logically consistent but extremely artificial and ambiguous, and contains many paradoxes, so that it must be glossed by introducing diverse conditions. It is evidently necessary to make further quests into the inconsistencies of this theory.

A radically new mode of measurement cannot be obtained by dealing with the existence probabilities of point-particles. It is the complete measurement of field quantities within a certain space-time domain that must be dealt with. This is connected with the many-time theory, originating in Dirac,(20) and discussed by Yukawa,(21) Tomonaga,(22) Watanabe(23) and Tanikawa,(24) and verges upon an attempt to reestablish the problem of measurement from a purely relativistic standpoint. But these trials stand little hope of further development as they possess no substantial content whatever.

Next comes the monistic field theory, which treats both field and matter with a single field equation, considering the singular points of the field as matter, and regarding matter as being perfectly derivable from the field equation. This was first done by Mie,(25) followed by Born(26) and Infeld,(27) and then by Bopp.(28) These must be said to be of perfectly substantialistic character. They are logically consistent, but too much emphasis is laid upon the divergence difficulties, and though these may be solved, the theories are too far removed from other experimental facts and give an impression of awkwardness. It might be said in their defence that this is because they penetrate further into the intrinsic nature of things than do other theories. But there remains the next decisive contradiction that in monistic field theory the only field capable of admittance is a Bose field, and therefore no possible combination of this type of field avails in creating matter with spin ½ and obeying Fermi statistics. Herein lies the most essential difficulty of monistic field theory. To conquer this, a considerably different method of attack would be required. Another difficulty is that an equation of motion for the singular points expressing matter does not always follow as a matter of course. Also, for instance, if some special significance is attached to the charge current density in Bopps theory, it would mean making a great departure from the standpoint of monistic field theory. This difficulty is just the opposite of that arising in the quantum mechanical treatment by considering mutual interaction. In that case spins of the value ½ and Fermi statistics could be introduced without any difficulty, but the structure of elementary particles did not come in, whereas in the case of monistic field theory, the structure of elementary particles is first introduced, but spin ½ and Fermi statistics are shut out. It is here that we can perceive the fundamental inconsistency of concepts.

Another problem confronting monistic field theory is the way in which to treat matter interacting with many kinds of fields Concerning this point studies are being made by the Nagoya University group under the direction of Dr. S. Sakata, and an interesting solution is already within sight in a different direction from monistic field theory; viz the mixed field theory.

Nextly, quantization is a great difficulty in the case of monistic field theory. It is found that the ordinary procedure of quantization is alien and quite unfitting to this standpoint. To carry out quantization, it is believed that there is no alternative but to proceed from the above mentioned standpoint of complete measurements of field quantities within a certain space-time domain. The peculiar method of quantization employed in Born's first paper[22] seems to be suggestive of pointing to the correct direction, though of course it is attended by serious mathematical difficulties.[23]

To recapitulate, monistic field theory must solve the problems of: spin and statistics; equation of motion for the singular points; finding an appropriate method of quantization.

We believe that it can be said from the above dialectic logical analysis that the conflict between the two fundamental concepts, matter and field is the contradiction in hand, which points out to us the direction in which we should proceed.

Institute of Physical and Chemical Research, Tokyo.

References

(1) Dirac: Proc. Roy. Soc., London (A) **167** (1938), 148, where the problem of a point-electron is solved classically, and the divergence separated. In connection with this singularity, too, a cohesive force has been considered. (Pryce: ibid **168** (1938), 389).

(2) Thursday-lecture at the Department of Physics, Osaka Imperial University.

(3) Heisenberg: Zeits. f. Phys. **110** (1938), 251.

(4) Landau: J. Phys. Acad. Sci. USSR, **2** (1940), 483.

(5) Heisenberg: Z. Phys. **113** (1939), 61.

(6) Bhabha: Proc. Roy. Soc., London **172** (1939), 384.

(7) Wentzel: Helv. Phys. Acta, **13** (1940), 269.

(8) Tomonaga: I.P.C.R. **39** (1941), 247; **40** (1942), 73.

(9) Heitler: Proc. Camb. Phil. Soc. **37** (1941), 291.

(10) Wilson: Pro. Camb. Phil. Soc. **3** (1941), 301.

(11) Christy & Kusaka: Phys. Rev. **59** (1941), 405.

(12) S. Tomonaga: I.P.C.R. **39** (1941), 247.

(13) S. Tomonaga: I.P.C.R. **40** (1942), 73.

(14) Watagin: ZS. f. Phys., **88** (1934), 92.

(15) Scherzer: Ann. d Phys. **34** (1939), 585.

(16) Markov: Journal of Phys. URSS, **2** (1940), 453.

(17) Dirac, Fock & Podolsky: Sow Phys. **2** (1932), 468.

(18) Whentzel: Zeit. f. Phys. **8**; (1933), 479, 635; **87** (1934), 726

(19) Pauli: Rev. of Mod. Phys. **15** (1943), 175. My sincere thanks are due to Drs. Tomonaga and Miyazima for discussing this paper with me.

(20) Dirac: Phys. Z. USSR, **3** (1933) 64.

(21) Yukawa: "Kagaku" **12** (1942), 251, 282, 322.

(22) Tomonaga: "Riken Iho" **22** (1943), 546, Prog. Theor. Phys. **1** (1946), 27.

(23) Watanabe: "Kagaku" **14** (1944), 102, Prog. Theor. Phys. **2** (1947), 71.

(24) Tanikawa: "Kagaku" **15** (1945), Prog. Theor. Phys. **1** (1946), 12.
(25) Mie: Ann. d. Phys **37** (1912), 511; **39** (1912). **85** (1028), 711.
(26) Born: Proc. Roy. Soc. London **143** (1933), 410.
(27) Born & Infield: Proc. Roy. Soc. London **144** (1934), 425; **147** (1934), 522; **150** (1935), 141.
(28) Bopp: Ann. d. Phys. **38** (1940), 345.
(29) Born: (26).
(30) Weyl: Phys. Rev. **46** (1934), 505

Progress of Theoretical Physics Vol. II, No. 4, Nov.~Dec., 1947.

On a Relativistically Invariant Formulation of the Quantum Theory of Wave Fields. III.

—Case of Interacting Electromagnetic and Electron Fields—

Zirô Koba, Takao Tati and Sin-ichirô Tomonaga.

(Received Nov. 6, 1946)

§ 5.* Derivation of the Ordinary Formalism.

We shall now show that the ordinary formalism of the quantum electrodynamics can be really derived from ours as the special case in which the surface C reduces into a plane parallel to the xyz–plane.

As was stated in I our fundamental equation (III) can be integrated by means of the unitary operator $T[C;\ C_0]$ defined by

$$T[C;\ C_0]=\prod_{C_0}^{C}\left(1-\frac{i}{\hbar}H_{I,II}(X)dV_X\right). \qquad (5\cdot1)$$

Namely, we obtain $\Psi[C]$ by giving its " initial " value $\Psi[C_0]$ in the form :

$$\Psi[C]=T[C;\ C_0]\,\Psi[C_0]. \qquad (5\cdot2)$$

We shall now transform each field quantity G by means of

$$G(X)\rightarrow T[C;\ C_0]^{-1}G(X)\,T[C;\ C_0] \qquad (5\cdot3)$$

and let $G[X;\ C]$ denote this transformed field. This field quantity depends, besides on x, y, z and t, on the variable surface C. The transformation (5·3) induces also the transformation of the generalized ψ–vector $\Psi[C]$:

$$\Psi[C]\rightarrow T[C;\ C_0]^{-1}\Psi[C]=\Psi[C_0]. \qquad (5\cdot4)$$

Then the transformed Ψ depends no longer on the surface C. In this way we obtain the " Heisenberg picture " of the behavior of our system, in

---

[*] §§ 1, 2, 3 and 4 have appeared in the forgoing issues of this journal : Progr. Theor. Phys ; 2 (1947), 101.

which the ψ-vector depends no longer on C, but the field quantities, on the other hand, depend on C, not only on t.

Also in our generalized Heisenberg picture the dependence of the field quantities on x, y, z and t are still given by the field equations (I) of the free fields. The dependences on C, on the other hand, are given by

$$\frac{\delta G[X;\, C]}{\delta C_P} = \frac{i}{\hbar}[H_{I,II}(P),\, G(X;\, C)] \tag{5·5}$$

as can be easily shown.

Now, the ordinary Heisenberg scheme of our system is obtained if one takes the variable plane C_T parallel to the xyz-plane whose intercept with the time axis is T. If the field quantities in this ordinary scheme are denoted by affixing the superscript o, they are given by

$$\overset{\circ}{G}(T,\, xyz) = G[t,\, xyz,\, C]_{\substack{t=T\\ C=C_T}}. \tag{5·6}$$

The time derivative of G with respect to the "common time" T is then given by

$$\begin{aligned}\frac{\partial \overset{\circ}{G}(T)}{\partial T} &= \left\{\frac{\partial G[t,C]}{\delta t} + \int_C \frac{\delta G}{\delta C_p} dF_p\right\}_{\substack{t=T\\ C=C_T}}\\ &= \left\{\frac{\partial G}{\partial t} + \frac{i}{\hbar}[\overline{H}_{I,II},\, G]\right\}_{\substack{t=T\\ C=C_T}}.\end{aligned} \tag{5·7}$$

If we denote by Div′, Curl′, etc. the differential operators in which the time differentiation is $\partial/\partial T$, and by Div, Curl, etc. those in which the time differentiation is $\partial/\partial t$, then we have the relations

$$\left.\begin{aligned}&\mathrm{Div}'\,\overset{\circ}{\boldsymbol{G}} = \left\{\mathrm{Div}\,\boldsymbol{G} - \frac{i}{\hbar}[\overline{H}_{I,II},\, G_0]\right\}_{\substack{t=T\\ C=C_T}}\\ &\mathrm{Curl}'\,\overset{\circ}{\boldsymbol{G}} = \begin{cases}\{\mathrm{Curl}\,\boldsymbol{G}\}_{t=T,\, C=C_T} & \text{for the space component,}\\ \left\{-\mathrm{grad}\,\overset{\circ}{G}_0 - \dfrac{\partial \vec{G}}{\partial t} + \dfrac{i}{\hbar}[\overline{H}_{I,II},\, \vec{G}]\right\}_{\substack{t=T\\ C=C_T}} & \text{for the time component.}\end{cases}\end{aligned}\right\} \tag{5·8}$$

Applying these formulas to the potential $\boldsymbol{A}$, noting that $\boldsymbol{A}$ is commutable with $\overline{H}_{I,II}$, we obtain

$$\left.\begin{aligned}\mathrm{Div}'\,\mathring{\boldsymbol{A}}&=\{\mathrm{Div}\,\boldsymbol{A}\}_{t=T,\ C=C_T}\\ \mathrm{Curl}'\,\mathring{\boldsymbol{A}}&=\{\mathrm{Curl}\,\boldsymbol{A}\}_{t=T,\ C=C_T}.\end{aligned}\right\}\tag{5·9}$$

Using the last relation, we have next

$$\mathrm{Div}'\ \mathrm{Curl}'\,\mathring{\boldsymbol{A}}=\mathrm{Div}'\{\mathrm{Curl}\,\mathring{\boldsymbol{A}}\}_{\substack{t=T\\ C=C_T}},$$

but applying (5·8), this is expressed as

$$=\left\{\mathrm{Div}\ \mathrm{Curl}\,\boldsymbol{A}+\frac{i}{\hbar}[\overline{H}_{I\,II},\ (\mathrm{Curl}\,\boldsymbol{A})_0]\right\}_{\substack{t=T\\ C=C_T}}$$

$$=\left\{\mathrm{Div}\ \mathrm{Curl}\,\boldsymbol{A}+\frac{i}{\hbar}\left[\overline{H}_{I,II},\ \left\langle-\frac{\partial\overset{\circ}{\vec{A}}}{\partial t}-\mathrm{Grad}\,A_0\right\rangle\right]\right\}_{\substack{t=T\\ C=C_T}}$$

$$=\mathrm{Div}\ \mathrm{Curl}\,\boldsymbol{A}+\varepsilon\langle\vec{J}\rangle\}_{t=T}$$

where the symbol $\langle\vec{a}\rangle$ means the four-vector whose time component vanishes and whose space component is $\vec{a}$. If we notice further the formula

$$\mathrm{Grad}\ \mathrm{Div}\,\boldsymbol{A}=\mathrm{Div}\ \mathrm{Curl}\,\boldsymbol{A}+\Box\boldsymbol{A},$$

we find

$$\mathrm{Div}'\ \mathrm{Curl}'\,\boldsymbol{A}=\{\mathrm{Grad}\ \mathrm{Div}\,\boldsymbol{A}-\Box\boldsymbol{A}+\varepsilon\langle J\rangle\}_{\substack{t=T\\ C=C_T}}$$

But, since $\boldsymbol{A}$ satisfies $\Box\boldsymbol{A}=0$, this last relation yields

$$\mathrm{Div}'\ \mathrm{Curl}'\,\mathring{\boldsymbol{A}}=\{\mathrm{Grad}\ \mathrm{Div}\,\boldsymbol{A}+\varepsilon\langle J\rangle\}_{\substack{t=T\\ C=C_T}}.\tag{5·10}$$

Now, the anxiliary condition (IV) requires, on the other hand,

$$\left\{\mathrm{Grad}\ \mathrm{Div}\,\boldsymbol{A}(P)+\varepsilon\,\mathrm{Grad}_P\int_C(\boldsymbol{J}(P'),\ \boldsymbol{N}(P'))\,D(P'-P)\,dF_{P'}\right\}\Psi=0.\tag{5·11}$$

Since the point P is here considered as lying on C, the integral can be carried out in the reference system whose space axes are tangent to C at P. As in this system $(\boldsymbol{J}(P')\,\boldsymbol{N}(P'))$ can be replaced simply by $-J_{\overline{n}}(P')$, and $\mathrm{Grad}_P\,D(P'-P)$ has the non-vanishing components $\delta(\vec{x}'-\vec{x})$ only in the $\overline{0}$-th direction, we obtain

$$\mathrm{Grad}_P \int_C (\boldsymbol{J}(P'),\ \boldsymbol{N}(P'))\, D(P'-P)\, dF_{P'}$$

$$=\begin{cases} -J_0(P) & \text{for the time component} \\ \ \ 0 & \text{for the space component.} \end{cases}$$

This gives, returning to the general reference system,

$$\mathrm{Grad}_P \int_C (\boldsymbol{J}(P'),\ \boldsymbol{N}(P'))\, D(P'-P)\, dF_{P'} = (\boldsymbol{J}(P),\ \boldsymbol{N}(P))\,\boldsymbol{N}(P)\,.$$

The auxiliary condition (5·11) is thus expressed in

$$\{\mathrm{Grad\ Div}\,\boldsymbol{A}(P)+\varepsilon\,(\boldsymbol{J}(P),\ \boldsymbol{N}(P))\,\boldsymbol{N}(P)\}\,\Psi=0\,,$$

which for $C=C_T$ reduces into

$$\{\mathrm{Grad\ Div}\,\boldsymbol{A}-\varepsilon\rangle J_0\langle\}\,\Psi=0 \tag{5·12}$$

where the symbol $\rangle a_0\langle$ means the four-vector whose space component is zero and whose time component is a_0.

We find in this way that, operated on Ψ, Grad Div $\boldsymbol{A}$ on the right-hand side of (5·10) can be replaced by $\varepsilon\rangle J_0\langle$, and this gives

$$\{\mathrm{Div}'\ \mathrm{Curl}'\,\mathring{\boldsymbol{A}}-\varepsilon\mathring{\boldsymbol{J}}\}\,\Psi=0 \tag{5·13}$$

the vectors $\langle\vec{J}\rangle$ and $\rangle J_0\langle$ making up the four-vector $\mathring{\boldsymbol{J}}$.

The last equation (5·13) corresponds just to the half set of the ordiary Maxwell equation:

$$\left.\begin{aligned} &\mathrm{curl}\,\overset{\circ}{\vec{H}}-\frac{\partial\overset{\circ}{\vec{E}}}{\partial T}=\varepsilon\overset{\circ}{\vec{J}} \\ &\mathrm{div}\,\overset{\circ}{\vec{E}}=\rho\,. \end{aligned}\right\}$$

The remaining set of equations

$$\left.\begin{aligned} &\mathrm{curl}\,\overset{\circ}{\vec{E}}+\frac{\partial\overset{\circ}{\vec{H}}}{\partial T}=0 \\ &\mathrm{div}\,\overset{\circ}{\vec{H}}=0 \end{aligned}\right\}$$

is trivial, since, expressed in terms of the potential, this is

$$\sum_{\text{cycl}} \text{Grad}' \ \text{Curl}' \overset{\circ}{\boldsymbol{A}}=0\ ,$$

which is satisfied identically. ($\sum\limits_{\text{cycl}}$ means cyclic sum)

§ 6. Elimination of the Auxiliary Condition.

We now carry out the elimination of the auxiliary condition by the method described in A.

First we expand the operator $\Xi_x[C]$ into the Fourier integral of the form :

$$\left.\begin{aligned} \Xi_x &= \frac{1}{(2\pi)^{3/2}}\int\left\{\Xi(\vec{k})\, e^{i(\boldsymbol{k};\boldsymbol{x})}+\Xi^{+}(\vec{k})\, e^{-i(\boldsymbol{k},\boldsymbol{x})}\right\}\frac{d\vec{k}}{k} \\ \Xi(\vec{k}) &= i[\boldsymbol{k},\boldsymbol{A}(\vec{k})]-\frac{\varepsilon}{2}\,\frac{i}{(2\pi)^{3/2}}\int_C (\boldsymbol{J}(P),\ \boldsymbol{N}(P))\, e^{-(\boldsymbol{k},\boldsymbol{x}_P)}\, dF_P \\ \Xi^{+}(\vec{k}) &= -i[\boldsymbol{k},\ \boldsymbol{A}^{+}(\vec{k})]+\frac{\varepsilon}{2}\,\frac{i}{(2\pi)^{3/2}}\int_C (\boldsymbol{J}(P)\, \boldsymbol{N}(P))\, e^{i(\boldsymbol{k},\boldsymbol{x}_P)}\, dF_P\,. \end{aligned}\right\} \tag{6 · 1}$$

As stated in A, we introduce the following four sets of coordinates and momenta(3)

$$\left.\begin{aligned} &R=(\boldsymbol{r},\ \boldsymbol{A}) && R^{+}=(\boldsymbol{r},\ \boldsymbol{A}^{+}) \\ &S=(\boldsymbol{s},\ \boldsymbol{A}) && S^{+}=(\boldsymbol{s},\ \boldsymbol{A}^{+}) \\ &K=(\boldsymbol{l},\ \boldsymbol{A}^{+}) && K^{+}=-(\boldsymbol{k},\ \boldsymbol{A}) \\ &L=(\boldsymbol{l},\ \boldsymbol{A}) && L^{+}=(\boldsymbol{k},\ \boldsymbol{A}^{+}) \end{aligned}\right\} \tag{6 · 2}$$

where $\boldsymbol{r}$, $\boldsymbol{s}$ and $\boldsymbol{l}$ are three four-vectors satisfying the relations

$$\left.\begin{aligned} &(\boldsymbol{r},\ \boldsymbol{r})=(\boldsymbol{s},\ \boldsymbol{s})=1, \quad (\boldsymbol{l},\ \boldsymbol{l})=0 \\ &(\boldsymbol{r},\ \boldsymbol{s})=0, \quad (\boldsymbol{r},\ \boldsymbol{k})=(\boldsymbol{s},\ \boldsymbol{k})=0 \\ &(\boldsymbol{r},\ \boldsymbol{l})=(\boldsymbol{s},\ \boldsymbol{l})=0, \quad (\boldsymbol{l},\ \boldsymbol{k})=1\ ; \end{aligned}\right\} \tag{6 · 3}$$

The vector $\boldsymbol{k}$ satisfys, of course, $(\boldsymbol{k},\ \boldsymbol{k})=0$, because it is the propagation vector of the electromagnetic wave.

---

(3) S. Hayakawa Y. Miyamoto and S. Tomonaga: J. Phys. Soc. Japan in press.

It is easily verified that, on account of (6·3), the commutation relations now take the following forms:

$$\left.\begin{aligned} &[R(\vec{k}),\ R^{+}(\vec{k}')]=[S(\vec{k}),\ S^{+}(\vec{k}')] \\ &=[K(\vec{k}),\ K^{+}(\vec{k}')]=[L(\vec{k}'),\ L^{+}(\vec{k}')]=+\frac{\hbar k}{2}\delta(\vec{k}-\vec{k}'), \\ &\text{other commutators}=0 \end{aligned}\right\} \qquad (6\cdot 4)$$

which show that our sets (6·2) represent really four canonically conjugate sets.

In terms of these variables the auxiliary condition is expressed by

$$\left.\begin{aligned} &\left(K^{+}+\frac{f}{2}\right)\Psi=0 \\ &\left(L^{+}-\frac{f^{*}}{2}\right)\Psi=0 \end{aligned}\right\} \qquad (6\cdot 5)$$

in which f stands for

$$f=\frac{\varepsilon}{(2\pi)^{3/2}}\int_{C}(\boldsymbol{J}(P'),\ \boldsymbol{N}(P'))\,e^{-i(\boldsymbol{k},\,\boldsymbol{x}_{P'})}\,dF_{P'}, \qquad (6\cdot 6)$$

and f^{*} is its conjugate complex quantity.

We introduce now the representation in which R, S, K and L are diagonal. Then the ψ-vector Ψ is represented by a functional of R, S, K, L and some variables describing the electron field. The auxiliary condition (6·5) in this representation takes the form:

$$\left.\begin{aligned} &\left(-\frac{\hbar k}{2}\frac{\delta}{\delta K}+\frac{f}{2}\right)\Psi=0 \\ &\left(-\frac{\hbar k}{2}\frac{\delta}{\delta L}-\frac{f^{*}}{2}\right)\Psi=0 \end{aligned}\right\} \qquad (6\cdot 7)$$

which can be satisfied identically by

$$\left.\begin{aligned} &\Psi=e^{\chi}\,\Omega \\ \text{with}\qquad &\chi=-\frac{1}{\hbar}\int(Lf^{*}-Kf)\frac{d\vec{k}}{k} \end{aligned}\right\} \qquad (6\cdot 8)$$

Ω being a functional not containing K and L.

It is now required to find the generalized Schrödinger equation for Ω.

We substitute thus (6·8) into (III), and calculate $\left(H_{I,II}+\frac{\hbar}{i}\frac{\delta}{\delta C_P}\right)e^{\chi}\Omega$.

Noting

$$\left.\begin{aligned} \boldsymbol{A}(x) &= \frac{1}{(2\pi)^{3/2}}\int\{\boldsymbol{A}(\vec{k})\,e^{i(k,x)}+\boldsymbol{A}^{+}(\vec{k})\,e^{-i(k,x)}\}\frac{d\vec{k}}{k} \\ \boldsymbol{A}(\vec{k}) &= R\boldsymbol{r}+S\boldsymbol{s}+L\boldsymbol{k}-K^{+}\boldsymbol{l} \\ \boldsymbol{A}^{+}(\vec{k}) &= R^{+}\boldsymbol{r}+S^{+}\boldsymbol{s}+K\boldsymbol{k}+L^{+}\boldsymbol{l}, \end{aligned}\right\} \tag{6·9}$$

we decompose $\boldsymbol{A}(X)$ into three parts:

$$\left.\begin{aligned} \boldsymbol{A}(X) &= \boldsymbol{A}_k(X)+\boldsymbol{A}_l(X)+\boldsymbol{B}(X) \\ \text{with}\qquad \boldsymbol{A}_k(X) &= \frac{1}{(2\pi)^{3/2}}\int\boldsymbol{k}(Le^{i(kx)}+Ke^{-i(k,x)})\frac{d\vec{k}}{k} \\ \boldsymbol{A}_l(X) &= \frac{1}{(2\pi)^{3/2}}\int\boldsymbol{l}(-K^{+}e^{i(kx)}+L^{+}e^{-i(kx)})\frac{d\vec{k}}{k} \\ \boldsymbol{B}(X) &= \frac{1}{(2\pi)^{3/2}}\int\boldsymbol{r}(Re^{i(k,x)}+R^{+}e^{-i(kx)})\frac{d\vec{k}}{k} \\ &\quad+\frac{1}{(2\pi)^{3/2}}\int\boldsymbol{s}(Se^{i(kx)}+S^{+}e^{-i(k,x)})\frac{d\vec{k}}{k} \end{aligned}\right\} \tag{6·10}$$

As discussed in A, two components $\boldsymbol{A}_k$ and $\boldsymbol{A}_l$ in (6·10) are to be regarded as describing the generalized longitudinal components, while $\boldsymbol{B}$ the generalized transverse component of the field. The energy $H_{I,II}=-\varepsilon(\boldsymbol{J},\boldsymbol{A})$ is divided also into three parts corresponding to the above decomposition of the field:

$$\left.\begin{aligned} H_{I,II} &= H_k+H_l+H_B \\ \text{with}\qquad H_k &= -\varepsilon(\boldsymbol{J},\ \boldsymbol{A}_k) \\ H_l &= -\varepsilon(\boldsymbol{J},\ \boldsymbol{A}_l) \\ H_B &= -\varepsilon(\boldsymbol{J},\ \boldsymbol{B}). \end{aligned}\right\} \tag{6·11}$$

We calculate now successively $H_k e^{\chi}\Omega$, $H_l e^{\chi}\Omega$ and $\frac{\hbar}{i}\frac{\delta}{\delta C_P}e^{\chi}\Omega$ in the following way:

(i) The calculation of $H_k(P)e^{\chi}\Omega$.

According to (6·10) and (6·11), H_k contains neither K^+ nor L^+. Further, the current density $\mathfrak{J}(P)$ in H_k commutes with the expression $(\mathfrak{J}(P'),\ \mathfrak{N}(P'))$ in f as can be shown by carrying out the integration in f in the reference system whose space axes are tangent to C at P. These two facts result in that $H_k(P)$ commutes with e^{χ}, so that we have

$$H_k e^{\chi}\Omega = e^{\chi} H_k \Omega \,. \tag{6·12}$$

(ii) The calculation of $H_l(P)e^{\chi}\Omega$.

$$\begin{aligned}
H_l e^{\chi}\Omega &= -\frac{\varepsilon\hbar}{(2\pi)^{3/2}}\int(\mathfrak{J},\ \mathfrak{l})\left\{\frac{ke^{i(\mathfrak{k},\mathfrak{x})}}{2}\frac{\delta}{\delta K}-\frac{ke^{-i(\mathfrak{k}\mathfrak{x})}}{2}\frac{\delta}{\delta L}\right\}\frac{d\vec{k}}{k}e^{\chi}\Omega \\
&= -\frac{\varepsilon\hbar}{(2\pi)^{3/2}}e^{\chi}\int(\mathfrak{J},\ \mathfrak{l}(\vec{k}))\left\{\frac{ke^{i(\mathfrak{k},\mathfrak{x})}}{2}\frac{\delta e^{\chi}}{\delta K}-\frac{ke^{-i(\mathfrak{k},\mathfrak{x})}}{2}\frac{\delta e^{\chi}}{\delta L}\right\}\frac{d\vec{k}}{k}\Omega \\
&= -\frac{\varepsilon}{(2\pi)^{3/2}}e^{\chi}\int(\mathfrak{J},\ \mathfrak{l}(\vec{k}))\left\{\frac{e^{i(\mathfrak{k},\mathfrak{x})}}{2}f(\vec{k})+\frac{e^{-i(\mathfrak{k},\mathfrak{x})}}{2}f^*(\vec{k})\right\}\frac{d\vec{k}}{k}\Omega \\
&= -e^{\chi}\frac{\varepsilon^2}{16\pi^3}\int(\mathfrak{J}(P),\ \mathfrak{l}(\vec{k}))\frac{d\vec{k}}{k}\int_C(\mathfrak{J}(P'),\ \mathfrak{N}(P'))\,\{e^{i(\mathfrak{k},\mathfrak{x}_P-\mathfrak{x}_{P'})} \\
&\qquad +e^{-i(\mathfrak{k},\mathfrak{x}_P-\mathfrak{x}_{P'})}\}\,dF_{P'}\,\Omega
\end{aligned}$$

If we put

$$\mathfrak{L}(X)=\frac{1}{16\pi^3}\int\mathfrak{l}(\vec{k})\,\{e^{i(\mathfrak{k},\mathfrak{x})}+e^{-i(\mathfrak{k},\mathfrak{x})}\}\frac{d\vec{k}}{k}\,, \tag{6 · 13}$$

we have, therefore,

$$H_l e^{\chi}\Omega = -e^{\chi}\varepsilon^2\int(\mathfrak{J}(P)\,\mathfrak{L}(P-P'))\,(\mathfrak{J}(P'),\ \mathfrak{N}(P'))\,dF_{P'}\,\Omega \tag{6·14}$$

(iii) The calculation of $\dfrac{\hbar}{i}\dfrac{\delta}{\delta C_p}e^{\chi}\Omega$.

We have first

$$\frac{\hbar}{i}\frac{\delta}{\delta C_P}e^{\chi}\Omega = e^{\chi}\left\{\frac{\hbar}{i}\frac{\delta\chi}{\delta C_P}+\frac{\hbar}{i}\frac{\delta}{\delta C_P}\right\}\Omega.$$

But we have

$$\frac{\hbar}{i}\frac{\delta\chi}{\delta C_P}=\frac{i\varepsilon}{(2\pi)^{3/2}}\int L(\vec{k})\frac{d\vec{k}}{k}\frac{\delta}{\delta C_P}\int_C(\boldsymbol{J}(P'),\ \boldsymbol{N}(P'))\,e^{i(\boldsymbol{k},\,\boldsymbol{x}_{P'})}\,dF_{P'}$$

$$-\frac{i\varepsilon}{(2\pi)^{3/2}}\int K(\vec{k})\frac{d\vec{k}}{k}\frac{\delta}{\delta C_P}\int_C(\boldsymbol{J}(P'),\ \boldsymbol{N}(P'))\,e^{-i(\boldsymbol{k},\,\boldsymbol{x}_{P'})}\,dF_{P'}\,.$$

If one uses here (3·7), we obtain

$$\frac{\hbar}{i}\frac{\delta\chi}{\delta C_P}=-\frac{i\varepsilon}{(2\pi)^{3/2}}\int L(\vec{k})\,\mathrm{Div}\,(\boldsymbol{J}(P)\,e^{i(\boldsymbol{k},\,\boldsymbol{x}_P)})\frac{d\vec{k}}{k}$$

$$+\frac{i\varepsilon}{(2\pi)^{3/2}}\int K(\vec{k})\,\mathrm{Div}(\boldsymbol{J}(P)\,e^{-i(\boldsymbol{k},\,\boldsymbol{x}_P)})\frac{d\vec{k}}{k}$$

$$=\frac{\varepsilon}{(2\pi)^{3/2}}\int(\boldsymbol{J}(P),\ \boldsymbol{k})\,L(\vec{k})\,e^{i(\boldsymbol{k},\,\boldsymbol{x}_P)}\frac{d\vec{k}}{k}$$

$$+\frac{\varepsilon}{(2\pi)^{3/2}}\int(\boldsymbol{J}(P),\ \boldsymbol{k})\,K(\vec{k})\,e^{-i(\boldsymbol{k},\,\boldsymbol{x}_P)}\frac{d\vec{k}}{k}$$

$$=\varepsilon(\boldsymbol{J}(P),\ \boldsymbol{A}^k(P))=-H_k\,. \tag{6·15}$$

Now summing up the results (i), (ii), and (iii), we obtain

$$\left(H_{I,II}(P)+\frac{\hbar}{i}\frac{\delta}{\delta C_P}\right)\Psi=\left(H_k(P)+H_l(P)+H_B(P)+\frac{\hbar}{i}\frac{\delta}{\delta C_P}\right)e^{\chi}\,\Omega$$

$$=e^{\chi}\left(H_B(P)+\varepsilon^2G(P)+\frac{\hbar}{i}\frac{\delta}{\delta C_P}\right)\Omega \tag{6·16}$$

where $G(P)$ is defined by

$$G(P)=-\int_C(\boldsymbol{J}(P),\ \boldsymbol{L}(P-P'))\,(\boldsymbol{J}(P'),\ \boldsymbol{N}(P'))\,dF_{P'}\,. \tag{6·17}$$

Multiplying (6·16) by $e^{-\chi}$ from the left we obtain the required equation for Ω:

$$\left\{H_B(P)+\varepsilon^2G(P)+\frac{\hbar}{i}\frac{\delta}{\delta C_P}\right\}\Omega=0 \tag{6·18}$$

in which the longitudinal parts of the field appear no longer, while it contains, on the other hand, the term $\varepsilon^2 G(P)$, which, according to (6·17), represents the effect on the electron field at the point P of the electron field at other points P'.

§ 7. Concluding Remarks.

We have thus shown that it is in fact possible to describe the behavior of the electromagnetic field interacting with electrons in a perfectly relativistic form according to our general scheme developed in I. But if we wish to apply similar methods to more general cases, for instance, to the cases of the meson field interacting with the electromagnetic field or the nucleon field, some generalization is necessary. In fact, in the case of the quantum electrodynamics the situation is exceptionally simple owing to the following two facts: (i) the interaction energy density $H_{I,II}(P)$ is a scalar function of the fields and (ii) the integrability condition $[H_{I,II}(P), H_{I,II}(P')]=0$ is satisfied from the beginning. These simplifying facts do no longer hold in the more general cases mentioned above.

However, if some generalization of the formalism is made, it is possible to develop the similar theory also in these more complicated cases as will be shown in the later paper.(4) The development of such a theory seems to us of interest, not only because we obtain thus formally more satisfactory theory, but also because in this way we can hope that some new aspects of the difficulties underlying the current quantum theory of the fields would reveal itself. Thus, for instance, the question arises: what bearing would have the so-called cut-off hypothesis on our theory?

In fact, Heisenberg(5) has thought that $H_{I,II}dV$ in the transformation functional $\prod_{C_0}^{C_1}\left(1-\frac{i}{\hbar}H_{I,II}dV\right)$ might be the suspicious element of the current theory. Miyazima(6) has once noticed that, although our theory seems at first sight to allow the introduction of a relativistically invariant cut-off process, taking as $H_{I,II}(P)$ not the energy density just at the world point P but some averrage over a finite world region surrounding P, such a procedure breaks necessarily the condition of integrability of the fundamental equation (III). Also it will be of great interest to examine Dirac's λ-process⁽⁷⁾ in the light of our theory. This process will be certainly incompatible with the integrability condition of the fundamental equation. In such a situation we must ask: is it possible to get over this difficully by some modification of the theory? Or, should we attribute physical meanings to the non-integrable Ψ? It is also possible that the defect of the current theory is more deeplying so that the formal feature of our theory will be maintained still in the future theory.

---

(4) S. Kanesawa and S. Tomonaga: Progr. Theor. Phys. in press.

(5) W. Heisenberg: ZS. Phys., 110 (1938), 251.

(6) T. Miyazima: Riken Ihô (Bull. I.P.C.R.), 23 (1944), 27.

Acknowledgement. This work was carried out at the Institute of Physics, Tokyo Bunrika-Daigaku. We wish to express our gratitude to our colleagues participating in the seminary of theoretical physics of the institute for the valuable discussions and helps they have given during the course of this work. Especially Mr. Hayakawa and Mr. Miyamoto have read our manuscripts and given many useful advices in making it.

Physics institutes, Tokyo University
and Tokyo Bunrika Daigaku.

---

(7) P. A. M. Dirac: Ann. Inst. Poincaré, 9 (1939), 13.

Progress of Theoretical Physics Vol. II, No. 4, Nov.~Dec. 1947.

On the Theory of Elementary Particles. I.*

Hideki YUKAWA.

(Received June 1, 1947)

§ 1. Introduction,

In the present theory of elementary particles, it is usual to start from the quantized field, which is equivalent to the assembly of indefinite number of one sort of elementary particles. The field which should be quantized is sometimes the one wellknown in classical theory as in the case of the electromagnetic field, but it may have sometimes no direct counterpart in classical theory as in the case of the electron field, which appears first in quantum mechanics as the wave function or the probability amplitude. Hitherto it has been regarded as one of the greatest success of relativistic quantum mechanics that the two things, i.e. the material particle and the field of force, which had been entirely different from each other in classical theory as well as in nonrelativistic quantum mechanics, could be coordinated by the method of second quantization. If, however, we want to go into the problem of interaction between two sorts of elementary particles, we find at once the essential difference still remaining between field and matter. For instance, we consider usually the interaction between two electrons as due to intermediary action of the electromagnetic field instead of direct interaction at distance. In this very assumption of the coexistence of the field around the point particle lies the origin of the wellknown difficulties of the present theory. Although recent development of the meson theory of nuclear forces seems to have made the situation more complicated because of the fact that the meson has been considered to be something with the dual nature, one of the main causes of newly added diffiulties can still be ascribed to the point interaction between the nucleon and the meson field. Under these circumstances we can follow either of two ways of revision and completion of the present theory of elementary particles. Namely, we can either pass over the difference between matter and field and devote ourselves to the establishment of the new scheme which may be applied to any kind of elementary particles, or we can first turn our attention to the difference of role played by field and matter and try to find the specified method of quantization appropriate for each of them, before we go into the construction of the general theory. Of course we cannot strictly distinguish between these two ways, but they are always mingled with each other in actual investigation. Nevertheless most of attempts hitherto made seem to lay stress on the revision of the method of quantization common to all sorts of elementary particles. In this paper, the author want to follow the latter way of discriminating field and matter in the begining. Thus we can show that a natural extention of the field concept, which has been so useful in physics for many years since, leads to a generalized mixed field theory, and further to a suggestion for the new method of quantization of the material field.†

---

* The content of this paper was read before the meetings of the Physical Society of Japan held at Kyoto University, Nov. 23, 1946 and at Nagoya University, March 1, 1947.

§ 2. An Extension of the Mixed Field Theory.

It is one of the well-established facts that the charged particle of spin 1/2 such as the electron or the proton has the neutral partner such as the neutrino or the neutron and various nuclear phenomena can be interpreted more easily by assuming the transition between charged and neutral states of an elementary particle such as the light particle or the nucleon than by consipering the charged and the neutral particles as entirely different entities. Thus, if we neglect the mass difference between charged and neutral states and assume for any one of such particles Dirac's wave equation of the form

$$\left(\sum_{\mu=0}^{3}\alpha_\mu p_\mu+\beta mc\right)\psi=0 \qquad (1)$$

in vacuum, where m is the rest mass and

$$\alpha_0=1,\quad \alpha_1=\rho_1\sigma_1,\quad \alpha_2=\rho_1\sigma_2,\quad \alpha_3=\rho_1\sigma_3,\quad \beta=\rho_3, \qquad (2)$$

---

† Detailed criticism of the concepts of field and matter in connection with the mixed field theory was also made by Sakata(1) and Taketani(2).

$$x_0=ct, \quad x_1=x, \quad x_2=y, \quad x_3=z \tag{3}$$

$$p_\mu=-i\hbar\frac{\partial}{\partial x_\mu}, \qquad (\mu=0,\ 1,\ 2,\ 3) \tag{4}$$

the wave function ψ, becomes a function of x_μ and σ_3, ρ_3 as well as τ_3, where τ_1, τ_2, τ_3 are isotopic spin variables, $\tau_3=\pm 1$ denoting neutral and charged states respectively. In the presence of other particles, it is usual to modify the operator in the wave equation by adding the term characterizing the interaction of the original particle with other particles. The most general form of this interaction term can be expressed as a certain function of all the variables for the original particle as well as those for the field originated by other particles. Such an operator can, however, be reduced always to a sum of linear functions of σ's and ρ's as well as of τ's, so that the wave equation takes the form

$$\left(\sum_{\mu=0}^{3}\alpha_\mu p_\mu+\beta mc+V\right)\psi=0 \tag{5}$$

with

$$V=\sum_{\lambda,\mu,\nu=0}^{3}\rho_\lambda\sigma_\mu\tau_\nu V_{\lambda\mu\nu} \tag{6}$$

where $\rho_0=\sigma_0=\tau_0=1$ and each of 64 coefficients $V_{\lambda\mu\nu}$ is a function of the particle variables x_μ and p_μ and of the field variables. (6) can alternatively be written in the form

$$\begin{aligned}V=(\tau_1+i\tau_2)\sum_{\lambda,\mu}\rho_\lambda\sigma_\mu V^{(1)}_{\lambda\mu}+(\tau_1-i\tau_2)\sum_{\lambda,\mu}\rho_\lambda\sigma_\mu V^{(2)}_{\lambda\mu}\\ +(\tau_0+\tau_3)\sum_{\lambda,\mu}\rho_\lambda\sigma_\mu V^{(3)}_{\lambda\mu}+(\tau_0-\tau_3)\sum_{\lambda,\mu}\rho_\lambda\sigma_\mu V^{(4)}_{\lambda\mu},\end{aligned} \tag{7}$$

in which the charged and the neutral fields are included as the first two term and the last two terms respectively. Each of these four terms can further be decomposed into scalar, vector, pseudoscalar and pseudovector fields in the usual manner, so that we arrive at the most general form of the mixed fields containing 16 sorts on the whole.

An obvious departure of this theory from the ordinary field theory is that the field in our case is not necessarily a function of x_μ alone, but may depend on p_μ also. Such an extension of the field concept was already attempted by Markow(3) in the case of the electromagnetic field in order to take into account the finite extent of the electron. In our theory such a sort of electromagnetic field can naturally be included in the last term of the expression (6) for V, other terms being interpreted as generalized meson fields of various types. It should be noticed, however, that the generalized field can include not only the electromagnetic field and various types of meson fields, but also the so-called pair fields such as the meson pair field and the electron-neutrino field. Therewith the distinction between the field theory in the narrow sense and the theory of action at distance such as proposed recently by Wheeler(4) becomes much more obscure.

§ 3. Equations for the Generalized Field.

In order to find the relations which exist between x_μ, p_μ and V's as the extension of the field equations, we consider first the simplest case of the electromagnetic field. In ordinary theory, the electromagnetic potentials A_ν are certain functions of x_μ alone, so that they can be represented in p-space by matrices of the form

$$\begin{aligned}(p_\mu' | A_\nu | p_\mu'') &= (2\pi\hbar)^{-4} \int \cdot\cdot \int e^{-ip_\mu' x_\mu'/\hbar} A_\nu(x_\mu') \delta(x_\mu' - x_\mu'') e^{ip_\mu'' x_\mu''/\hbar} \\ &\times (dx_\mu')^4 (dx_\mu'')^4 = (2\pi\hbar)^{-4} \int \cdot\cdot \int e^{-i\pi_\mu' x_\mu'/\hbar} A_\nu(x_\mu') (dx_\mu')^4 \qquad (8)\end{aligned}$$

and conversely we have

$$A_\nu(x_\mu') = \int \cdot\cdot \int (p_\mu' | A_\nu' | p_\mu'') e^{i\pi_\mu' x_\mu'/\hbar} (d\pi_\mu')^4, \qquad (9)$$

where

$$\pi_\mu' = p_\mu' - p_\mu''. \qquad (10)$$

Now we take into account another well-established act that the transition of an elementary particle from one charged state to another is accompanied by the emission or absorption of a photon with the rest mass zero, the energy and the momentum satisfying the conservation laws. In other words the electromagnetic field in vacuum, when quantized, should be equivalent to an assembly of photons, so that the matrices $(p_\mu' | A_\nu' | p_\nu'')$ have the form

$$(p_\mu' | A_\nu | p_\mu'') = A_\nu(\pi_\mu') \delta(\pi_\mu' \pi^{\mu'}) \qquad (11)$$

where

$$\pi_\mu'\pi^{\mu'}=-(\pi_0')^2+(\pi_1')^2+(\pi_2')^2+(\pi_3')^2 \qquad (12)$$

We can extend the expression (11) without missing the expectation that the generalized electromagnetic field in vacuum will still represent in some way an assembly of photons, when it is quantized. We assume namely a more general form

$$(p_\mu'|A_\nu|p_\mu'')=(p_\mu'|A_\nu'|p_\mu'')\,\delta(\pi_\mu'\pi^{\mu'}) \qquad (13)$$

where the matrix elements $(p_\mu'|A_\nu'|p_\mu'')$ are not necessarily functions of π_μ' alone, corresponding to the situation that the potentials A_ν are not functions of x_μ alone.

Alternatively we may start from the D'Alembertian for the ordinary electromagnetic potential

$$\Box A_\nu\equiv\left(-\frac{\partial^2}{\partial x_0^2}+\frac{\partial^2}{\partial x_1^2}+\frac{\partial^2}{\partial x_2^2}+\frac{\partial^2}{\partial x_3^2}\right)A_\nu(x_\lambda)=0 \qquad (14)$$

instead of the expression (9). In p-space, the equations (14) are represented in the form

$$\int\cdots\int e^{-ip_\lambda'x_\lambda'/\hbar}\{\Box A_\nu(x_\lambda')\}e^{ip_\lambda''x_\lambda'/\hbar}(dx_\lambda')^4=0 \qquad (15)$$

or

$$\int\cdots\int e^{-i(p_\lambda'-p_\lambda'')x_\lambda'/\hbar}(p_\mu'-p_\mu'')(p^{\mu'}-p^{\mu''})A_\nu(x_\lambda')(dx_\lambda')^4$$
$$=(p_\lambda'|p_\mu(p^\mu A_\nu-A_\nu p^\mu)|p_\lambda'')-(p_\lambda'|(p^\mu A_\nu-A_\nu p^\mu)p_\mu|p_\lambda'')=0, \quad (16)$$

which is equivalent to the operator equations

$$[p_\mu,\,[p^\mu,\,A_\nu]]=0. \qquad (17)$$

Now, if we assume the potential operators A_ν for the generalized electromagnetic field in vacuum to satisfy the equations of the same form (17), the generalized potential will be given by the expression (13). Further, if we define the generalized electromagnetic field tensor by

$$-i\hbar F_{\mu\nu}=[p_\mu,\,A_\nu]-[p_\nu,\,A_\mu], \qquad (18)$$

we obtain operator equations of the Maxwell type

$$[p^\mu,\,F_{\mu\nu}]=0, \qquad (19)$$

provided that the generalized Lorentz condition

$$[p^{\mu}, A_{\mu}]=0. \tag{20}$$

holds for A_{μ}.

Above considerations can be extended easily to the case of the field, which is accompanied by particles of finite rest mass μ. In the case of the vector field, for example, the field equations in vaccum can be transformed to simultaneous operator equations

$$\left.\begin{aligned}[p_{\mu}, A_{\nu}]-[p_{\nu}, A_{\mu}]&=-iF_{\mu\nu}\\ [p_{\mu}, F_{\mu\nu}]&=-i\mu^2c^2A_{\nu},\end{aligned}\right\} \tag{21}$$

where A_{ν}, $F_{\mu\nu}$ are four and six vector operators respectively.

§4. A Suggestion for the Quantization of Generalized Field.

In the ordinary theory, field equations in the presence of source particles are obtained by adding, for example, $4\pi i\hbar$ times a four vector s_{ν} representing the charge and current densities of the particles to the right hand side of (19). In order to quantize the system consisting of the field and the particles, it is usual to consider the field variables $A_{\nu}(x_{\lambda})$, $F_{\mu\nu}(x_{\lambda})$ as operators satisfying appropriate commutation relation between each pair of them and also to consider the wave function ψ and its complex conjugate $\tilde{\psi}$ as operators satisfying commutation relations of Fermi-Dirac type, whereas the charge and current densities are expressed by a four vector operator

$$s_{\mu}=-e\tilde{\psi}a_{\mu}\psi\,. \tag{22}$$

Now in the case of the generalized field theory, the left hand side of (19) is no more diagonal in x-space, so that the four vector on the right hand side should take instead of (22) a more general form

$$(x_{\lambda}'\,|\,s_{\nu}\,|\,x_{\lambda}'')=-e\tilde{\psi}(x_{\lambda}'')a_{\mu}\psi(x_{\lambda}'). \tag{23}$$

It is interesting to recollect that Heisenberg⁽⁵⁾ introduced such a sort of density operator as the quantized form for Dirac's density matrix in the theory of the positron.⁽⁶⁾ In this connection, it should be noticed that there occurs in nature no process, in which only one parlicle with spin 1/2 such as a nucleon is created or annihilated, but always a pair of such particles is created or annihilated simultaneously. Thus the operators ψ, ψ have by themselves no direct correspondence with the natural processes, so that it seems more appropriate to adopt a formalism, in which only bilinear combinations of ψ, $\tilde{\psi}$ sucn as (23) appear. Detailed accounts as to actual method of quantization on these lines, however, will be given in part II of this paper.

Physics Department, Kyoto University.

References

(1) Sakata, Prog. Theor. Phys. **2** (1947), 145.
(2) Taketani, ibid. 187.
(3) Markow, Jour. Phys. **2** (1940), 453.
(4) Wheeler and Feyman, Rev. Mod. Phys. **17** (1945), 157.
(5) Heisenberg, Zs. f. Phys. **90** (1934), 209.
(6) Dirac, Proc. Cambr. Phil. Soc. **30** (1934), 150.
(7) Snyder, Phys. Rev. **71** (1947), 38; **72** (1947), 68.

Note added in proof; very recently Snyder(7) arrived at the same form of operator equations for the electromagnetic field as (17) and (19).

Letters to the Editor

Correction due to the Reaction of "Cohesive Force Field" for the Elastic Scattering of an Electron.

Recently Sakata and Hara[1)] introduced a scalar neutral cohesive force field (the so-called C-meson field) which is assumed to interact with charged particles, and by imposing a relation

$$f^2=2e^2 \qquad (1)$$

on the coupling constants e, f of the latters with electromagnetic and cohesive field respectively, they succeeded in obtaining a non-diverging value for the self-energy of an electron and also suggested a possibility of explaining the mass difference between proton and neutron.

Thus it will be worth while to examine whether or not this promising theory is able to eliminate other divergencies in the quantum electrodynamics, at least those appearing in the same approximation. We have taken up as a simple example the elastic scattering of an electron and estimated the effect of field reactions on this phenomena.

A perturbation calculation has been carried out on a similar line as Dancoff[2)] up to the order of e^2 and f^2, all the relativistic effects being thereby taken into account. The result is:

$$\frac{\delta\sigma}{\sigma_0}\sim\frac{1}{12\pi}\,\frac{(p-q)^2}{E^2+(pq)+m^2}(14e^2-9f^2)$$
$$\times\int\frac{dk}{k}+\text{finite terms} \qquad (2)$$

where σ_0 is the uncorrected cross-section of the elastic scattering, $\delta\sigma$ its correction due to the field reactions, p and q the momentum of the incident and the scattered electron respectively, E the energy and m the rest mass of the electron. It is seen that, in this case also, the divergencies due to both fields have the same form with different signs, so that it is possible that the infinities cancel out if the coupling constants have suitable values. For this, however, the relation

$$f^2=\frac{7}{9}\cdot 2e^2 \qquad (3)$$

is required which does not agree with (1).

In the above calculation we have assumed that the scattering potential is the fourth component of a four vector. Now the fundamental idea of the cohesive force theory is that the electromagnetic field is always accompanied by the C-meson field, so that it will be more consistent to introduce a scattering potential of the four-vector type.[3)] But the inclusion of such a potential makes the result more unsatisfactory: the condition which must be imposed on the coupling constants so as to bring a convergent result must be different for different values of the momenta of the initial and the final state of the electron.

It may be remarked that the divergence of the scattering processes by the polarization charge and durrent of the vacuum, which the existence of the scattering potential induces, have been subtracted (otherwise the result would have been still worse). But such a procedure is always shadowed by the ambiguity of the existing positron theory, the theory of "C-meson" giving no unique prescription at this point.

Summing up these considerations, it may not be too presumptious to conclude: The present formalism of the "cohesive force field" theory is not sufficient to eliminate all the difficulties of the quantum electrodynamics.

A fuller account will appear in a later issue of this journal.

D. Ito, Z. Koba and S. Tomonaga.

Physics Institute, Tokyo University and
Tokyo Bunrika Daigaku.

Nov. 1, 1947

References

1) S. Sakata and O. Hara: Progr. Theor. Phys., 2 (1949), 30.
2) S. M. Dancoff: Phys. Rev., 55 (1939), 959.
3) The necessity of the introduction of a world-scaler potential was first noticed by Mr. Miyamoto.

Errata to the Above Letter.

In a previous letter we have reported that, when the reactions of electromagnetic field as well as cohesive force field (C-meson field introduced by Sakata and Hara[1]) are simultaneously taken into account, the correction to the cross-section of the elastic scattering of an electron by a fixed electrostatic potential turns out

$$\frac{\partial\sigma}{\sigma_0}\cong\frac{m^2}{12\pi}\,\frac{(p-q)^2}{E^2\{E^2+(pq)+m\}^2}\times(14e^2-9f^2)\int^{\infty}\frac{kd}{k} \qquad (1)$$

(as to the notations refer to the previous letter), and, therefore, the condition

$$f^2=2e^2 \qquad (2)$$

imposed on the coupling constants in order to make the self-energy of an electron finite cannot eliminate the divergence appearing in (1). We have thus concluded that the present formalism of the *C*-meson theory might be still insufficient —.

But in re-examining our calculation, we found processes which had not been considered so far, the contribution of which alter the situation completely. A process which had been missed in our calculation (as well as in the similar calculation of Dancoff[2]) was, for instance, the following one: in the intermediate state there appears, besides the incident electron p a virtual pair q and $(-q)^+$ (+ means a positron) which has been created through the Coulomb interaction of a vacuum electron upon itself (the matrix element for this process being infinity); the scattering potential annihilates then the pair p and $(-q)^+$, so that the final state with the scattered electron q is attained. (In Dancoff's paper[2] and our letter the corresponding processes through emission and re-absorbtion of a transverse photon alone had been taken into account and had yielded the fatal divergence). This new process, together with a similar one, contributes to the cross sections as

$$\frac{\partial\sigma}{\sigma e}\cong\frac{e^2m^2}{3\pi}\,\frac{(p-q)^2}{E^2\{E^2+(pq)+m^2\}}\times\int^{\infty}\frac{dk}{k}\,. \qquad (3)$$

When this term is added to (1), the total correction does really converge if the coupling constants satisfy the relation (2). The same situation occurs also when the sattering potential is a world scalar.

Our conclusion must be, therefore, altered into the following: *The theory of cohesive force fields effective also to eliminate the divergence difficulty appearing in the scattering problem in the second order approximation. The divergence due to the vacuum polarization effects, however, cannot be eliminated by introducing this new field. At this point we must resort to the subtraction hypothesis of the positron theory.*

A fuller account of our corrected calculation will appear in a later issue of this journal. The comparison of this result with that obtained by the "self-consistent" subtraction method recently proposed by one of us[3] is discussed elsewhere.[4] Other related problems are now being investigated.

D. Ito, Z. Koba and S. Tomonaga

Physics Institutes, Tyoto University
and Tokyo Bunrika Daigaku.

Dec, 30, 1947.

References

1) S. Sakata and O. Hara: Progr. Theor. Phys. 2 (1947), 30.
2) S. M. Dancoff: Phys. Rev. 55 (1939), 959.
3) T. Tati and S. Tomonaga: Lecture at the symposium on the theory of elementary particles, Nov. 1947. Progr. Theor. Phys., 2 (1947).
4) Z. Koba and S. Tomonaga: See the next letter.

Application of the "Self-Consistent" Subtraction Method to the Elastic Scattering of an Electron

The experimental evidence of the level shift of hydrogen atom[1] and its theoretical explanation by Bethe[2] have thrown a new light upon the problem concerning the reaction of the electromagnetic field. One of us has proposed in connection with it a "self-consistent" subtraction method.[3] The Hamilton function describing the electromagnetic and electronic fields in interaction

$$H=H_{\mathrm{electron}}+H_{\mathrm{radiation}}+H_{\mathrm{interaction}} \qquad (1)$$

is re-written in the form

$$H=\left(H_{\mathrm{electron}}+\delta m\int\psi^*\beta\psi dv\right)+H_{\mathrm{radiation}} + \left(H_{\mathrm{interaction}}-\delta m\int\psi^*\beta\psi dv\right) \qquad (2)$$

where δm is taken in such a way that the mass correction of an electron due to the perturbation vanishes when one regards $(H_{\mathrm{electron}}+\delta m\int\psi^*\beta\psi dv)$ as describing free electrons, while in the perturbation energy is included an additional term $(-\delta m\int\psi^*\beta\psi dv)$. It is clear that for this purpose we should put for δm the electromagnetic mass correction in the ordinary sence;[4]

$$\delta m=\frac{3mc^2}{2\pi}\int^{\infty}\frac{dk}{k}, \qquad (3)$$

but with the re-interpretation of m standing on the right-hand side: in this m the mass correction is already included, because we have added the corresponding term in the Hamiltonian for the free electron, regarding that the observed mass is just of this corrected value.

Now let us consider the elastic scattering of an electron by a fixed electrostatic potential. Once Dancoff[5] calculated the correction to the cross-section for this process due to the reaction of the electromagnetic field and his result has been recently rectified by our more detailed consideration into

$$\frac{\delta\sigma}{\sigma_0}\simeq\frac{3e^2m^2}{2\pi E^3}\,\frac{(p-q)^2}{E_2+(pq)+m^2}\int^{\infty}\frac{dk}{k} \qquad (4)$$

(as to notations refer to our previous letters).

Here we apply our new interaction Hamiltonian $(H_{\mathrm{interaction}}-\delta m\int\psi^*\beta\psi dv)$ to this problem. Then the "counter-self-energy" term $(-\delta m\int\psi^*\beta\psi dv)$ gives rise to an additional correction, which just cancels the divergence in (4) of the previous result and thus one can expect a reasonable value for the total correction.[7]

As was reported lately,[6] a finite but perhaps slitely different value is obtained on the C-meson hypothesis. This is due to the fact that, in the case of C-mesonic field alone interaction with electron, the divergence is also eliminated by the contribution of the counter-self-energy term. Such a situation indicates an intimate inner relation to exist between the reaction effects in two cases: self-energy on one hand and correction to the elastic scattering process on the other. This point will be more clarified by the relativistic treatment of the self-consistent subtraction procedure in the super-many-time[3),8)] formulation.

The similar treatment of the vacuum polarization effect and other related problems are now being investigated. A fuller account will appear in a later issue of this journal.

Z. Koba and S. Tomonaga.

Physics Institutes, Tokyo University
and Tokyo Bunrika Daigaku.
Dec. 30, 1947.

References

1) W. E. Lamb and R. C. Retherford: Phys. Rev. 72 (1947), 241.
2) H. A. Bethe: Phys. Rev., 72 (1947), 339.
3) T. Tati and S. Tomonaga; Lecture at the symposium on the theory of elementary particles. Nov. 1947. Progr. Theor. Phys., 2 (1947).
4) V. Weisskopf: Phys. Rev., 56 (1936), 72.
5) S. M. Dancoff: Phys. Rev., 55 (1939), 959.
6) D. Ito, Z. Koba and S. Tomonaga: Progr. Theor. Phys., 2 (1947).
7) Now being calculated by our colleagues.
8) T. Tati and S. Tomonaga: to be published in this journal.

The Supplementary Note to the Generalized Transformation Fuction*

Recentry Dirac[1] has shown that an analogy between the classical and quantum contact transformation is established by defining a general function of non-commuting observables. Dirac's defintion however is too formal, it seems to be doubtfull whether or not really there exists such generel functions of non-commuting observables except when the observables are well-ordered.

In this note, we show that, according to the theory of the generalized transformation operator (g.t.o.) which was presented by the author,[2] the general functions of non-commuting observables are really defined and the same analogy between classical and quantum mechanics as Dirac is obtained. The g. t. o. is defined by

$$T(q_T'q_t')\doteq Mq_T'D(T-t)Mq_t',$$
$$D(T-t)=\exp\left\{-\frac{i}{\hbar}H\cdot(T-t)\right\} \qquad (1)$$

where Mq_t' and Mq_T' denote the observation-operators belonging to such observations that we obtain the result q_t' and q_T' when a observation of q are made at time t and T, respectivery, H is the Hamiltonian of the system. Corresponding to the g.t.f., here we define a generalized momentum operator (g.m.o.)

$$P_t(q_T'q_t')=Mq_T'D(T-t)P_tMq_t' \qquad (2)$$

and

$$P_T(q_T'q_t')=Mq_T'P_TD(T-t)Mq_t' \qquad (3)$$

where p's are conjugate momenta to q's.

Now, the mixed representation of P_T

$$(q_T'|P_T|q_t')=\sum_{q_T''}(q_T'|P_T|q_T'')$$
$$\times(q_T''|D(T-t)|q_t')=i\hbar\frac{\partial}{\partial q_T'}$$
$$\times(q_T'|D(T-t)|q_t')$$
$$=(q_t'|\,i\hbar\,\frac{\partial}{\partial q_T'}T(q_T'q_t')|q_t'),$$

then we obtanin

$$P_T(q_T'q_t')=i\hbar\frac{\partial}{\partial q_T'}T(q_T'q_t'), \qquad (4)$$

similarlly

$$P_t(q_T'q_t')=-i\hbar\frac{\partial}{\partial q_t'}T(q_T'q_t'). \qquad (5)$$

Thus we see that the g.t.o. $T(q_T'q_t')$ corresponds to the generating function of classical contact transformation.

In the case when $T(q_T'q_t')$ vanishes for some values of q_t' and q_T', we can choose the functions of q_t' and q_T', $f_i(q_T'q_t')$ $(i=1, 2,\dots n)$ which vanish everywhere except where $T(q_T'q_t')$ vanishes, and then

$$F_i(q_T'q_t')=f_i(q_T'q_t')T(q_T'q_t')=0$$
$$(i=1,2,\dots n) \qquad (6)$$

everywhere, the equations (6) give the additional conditions of the observables q_t and q_T.

We can again set up an analogy with the classical contact transformation

$$\left.\begin{aligned}P_T(q_T'q_t')&=i\hbar\frac{\partial T(q_T'q_t')}{\partial q_T'}\\&\quad+\sum_i i\hbar\lambda_i\frac{\partial F_i(q_T'q_t')}{\partial q_T'}\\P_t(q_T'q_t')&=-i\hbar\frac{\partial T(q_T'q_t')}{\partial q_t'}\\&\quad-\sum_i i\hbar\lambda_i\frac{\partial F_i(q_T'q_t')}{\partial q_t'}\end{aligned}\right\} \qquad (7)$$

Our theory now enables us to set up the general function of non-commuting observables which fall into the order of the times to which they refer, for example, q_T, q_t, $T>t$. We may define the general function of q_T, q_t as

$$F(q_T'q_t')=f(q_T'q_t')T(q_T'q_t') \qquad (8)$$

where $f(q_T'q_t')$ is a function of the eigenvalues q_T' and q_t'.

A generalization to the cases of more observables than two and in which q_T and q_t are different observables is straightfoward, namely

$$F(q'_{t_n},\, q'_{t_{n-1}},\, \dots,\, q'_{t_2},\, q'_{t_1})$$
$$=f(q'_{t_n},\, q'_{t_{n-1}},\, \dots,\, q'_{t_2},\, q'_{t_1})$$
$$\times T(q'_{t_n},\, q'_{t_{n-1}},\, \dots,\, q'_{t_2},\, q'_{t_1}) \qquad (9)$$
$$t_1<t_2<\ \dots<t_{n-1}<t_n$$

$$T(q'_{t_n} q'_{t_{n-1}} \cdots q'_{t_2} q'_{t_1})$$
$$= M_{q'_{t_n}} D(t_n - t_{n-1}) M_{q'_{t_{n-1}}} \cdots$$
$$M_{q'_{t_2}} D(t_2 - t_1) M_{q'_{t_1}} \quad (10)$$

In conclusion we wish to express our cordial thanks to Prof. H. Yukawa and Prof. S Sakata for their kind interest in this work.

Y. TANIKAWA

Institute of Theoretical Physics,
Nagoya University.
May 15, 1947.

References

1) P. Dirac: Rev. Mod. Phys., 17 (1945), 195.
2) Y. Tanikawa: Prog. of Theor. Phys. 1 (1947), 12.

* The content of this note was read before the Kansai-Branch Meeting of the Japan Physical Society held at Nagoya University on Mar. 1, 1947.

On the Cosmic-Ray Meson and the Nuclear Meson

It seems to be believed that there are several quantitative discrepancies between the properties of the cosmic-ray meson observed in cosmic-ray experiments and those of the nuclear meson predicted in Yukawa's theory. These contradictions have not been fully solved until now, in spite of many theoretical efforts.

In April, 1942, the present author put foward the hypothesis that the cosmic-ray meson is not directly related to the nuclear meson, but is produced from the decay of the nuclear meson whose properties are given by Yukawa's orginal theory, and the former is a scalar Bose particle.(1) After a short time, Sakata adopted another alternative that the cosmic-ray meson should be distinguished from the nuclear one on the same lines, but it is a Fermi particle of spin one-half.(2)

At that earlier time, there being no clear experimental evidence favorable for our hypothesis, the theory was not fully developed. Recently, in connection with the experiments on the negative meson capture by the Rome Group,(3) hypothesis similar to ours which distinguish between the cosmic-ray and the nuclear meson have been put forward by American authors.(4) These differ from one another only in the particular combination of the two sorts of mesons. The combination is formed of Fermi or Bose particles for the cosmic-ray and the nuclear meson respectively. But the unique selection between these combinations can not yet be decided from the present available experiments. Our hypothesis also seems to be worth presenting here again. If we denote the nuclear meson and the cosmic-ray meson by Y and M respectively, the simplest interaction between charged $Y^\pm$ and $M^\pm$ is given by

$$Y^\pm \rightleftarrows M^\pm \qquad (1)$$

which is similar to that between the different components of Dirac's electron, and whose interaction magnitude is given by a dimensionless constant γ. The vector nuclear meson $Y^\pm$ can disintegrate into the scalar cosmic-ray meson $M^\pm$ and a photon, if the mass of the former is largar than that of the latter:

$$Y^\pm \rightarrow M^\pm + \nu \qquad (2)$$

The decay formula is given as follows:

$$\frac{1}{\tau_0} = \frac{\gamma^2}{96} \frac{e^2}{\hbar c} \frac{\mu c^2}{\hbar} f(x) \qquad (3)$$

where μ is the nuclear meson mass, $f(x)$ is a rather complicated function of the mass ratio x of the two mesons. If we put $x = \frac{1}{2}$, $\gamma = 10^{-6}$ we obtain

$$\tau_0 \simeq 10^{-8} \text{ second.} \qquad (4)^{(5)}$$

The interaction between M and the nucleon can take place through a virtual transformation of M into Y, even if a direct interaction between them is not assumed. As γ is very small, the interaction of M with the nucleon is also small. The K-orbital M^- capture therefore is smaller by a factor 10^{-12} than the corresponding Y^--capture. This value is compatible with the Rome group experiments. If we assume a direct interaction between M and the nucleon, the magnitude must be of the same order as that of the interaction M with the light particles which will be given by the natural decay of M^- into electron-neutrino observed in the cosmic ray.(6)

The detailed account in a slightly more enlarged form than the original one will be given as soon as possible in a later issue of this journal.

Y. Tanikawa

Institute of Theoretical Physics,
Nagoya University.
Nov. 24, 1947.

References

1) Private conversations with Prof. H. Yukawa and Prof. S. Sakata at the Institute of Theoretical Physics, Kyoto University. The contents were read before the Annual Meeting of I. P. C. R. in June, 1942.
2) S. Sakata and T. Inoue: Progress of Theoretical Physics **1** (1946) 143.
3) M. Conversi, E. Pancini and O. Pinccioni: Phys. Rev. **71** (1947) 209.
4) R. E. Marshak and H. A. Bethe: Phys. Rev **72** (1947) 506.
B. Pontecorvo: Phys. Rev. **72** (1947) 246.
5) R. E. Marshak and H. A. Bethe: l. c.
6) G. Gamow: Phys. Rev. **71** (1947) 320.

INDEX TO VOLUME 2

Araki–G. Quantum Mechanics of Circularly Polarized Photons. II. Application to the Normal Zeeman Effect. 1

Araki–G. On the Modification of the Hamiltonian of a Nuclear Particle due to its Virtual Dissociation ....89 (L)

Hara–O. (See Sakata–S.).........30 (L)

Hayakawa–S. and Tomonaga–S. On the Theory of Air Schower initilated by Secondary Soft Rays. ....162 (L)

Hayashi–C. Giant Stars Producing Energy by C–N Reactions. 127

Ito–D., Koba–Z. and Tomonaga–S. Correction to the Reaction of " Cohesive Force Field " for the Elastic Scattering of an Electron. .... 216 (L)

Ito–D., Koba–Z. and Tomonaga–S. Errata to the Above Letter .... 217 (L)

Kanai–E. Some Remarks on the Non-Infinitesimal Commutation Relations. 136

Kanai–E., Kobayasi–M. and Kokaji–S. On the Electromagnetic Properties of Nucleon. II. 151

Kanai–E., Kobayasi–M. On the Electromagnetic Properties of Nucleon. III. 179

Koba–Z. (See Ito–D.).........216 (L)

Koba–Z. (See Ito–D.........217 (L)

Koba–Z., Tati–T. and Tomonaga–S. On a Relativistically Invariant Formulation of the Quantum Theory of Wave Field. II. 101

Koba–Z., Tati–T. and Tomonaga–S. On a Relativistically Invariant Formulation of the Quantum Theory of Wave Field. III. 198

Koba–Z. and Tomonaga–S. Application of the " Self-Consistent " Subtraction Method to the Elastic Scattoring of an Electron ....218 (L)

Kobayasi–M. (See Kanai–E.).........151, 179

Kokaji–S. (See Kanai–E.).........151

Miyahara–S. On the Theory of Ferromagnetism at 0°K. 25

Nakabayasi–K. On the Mesic Self-Field around a Nucleon. 163

Nakamura–S. and Ono–K. On the β–Ray Spectrum of Cu^{64}. ....158 (L)

Noma–S. Modified Meson Pair Theory. ....159 (L)

Ono–K. (See Nakamura–S.).........158 (L)
Sakata–S. The Theory of the Interaction of Elementary Particles. 145
Sakata–S. and Hara–O. The Self-Energy of the Electron and Mass Difference of Nucleons. ... .30 (L)
Takagi–S. (See Yamamoto–T.).........160 (L)
Taketani–M. Conflict between Matter and Field and Analisis of the Difficulties of the Theory of Elementary Particles. 187
Tanikawa–Y. On the Lifetime of the Nutral Meson. ....31 (L)
Tanikawa–Y. On the Decay of the Charged Meson. ....31 (L)
Tanikawa–Y. The Supplementary Note to the Generalized Fransformation Function.........219 (L)
Tanikawa–Y. On the Cosmic-Ray Meson and the Nuclear Meson. ....220 (L)
Tati–T. (See Koba–Z.).........101, 198
Tati–T. (See Koba–Z.).........198
Tomonaga–S. On the Effect of the Field Reactions on the Interaction of Mesotron and Nuclear Particles. III. 6
Tomonaga–S. On the Effect of the Field Reactions on the Interaction of Mesotron and Nuclear Particles. IV. 63
Tomonaga–S. Some Considerations about the Production of Meson Schowers in Matter. ....90 (L)
Tomonaga–S. (See Hayakawa–S.).........162 (L)
Tomonaga–S. (See Ito–D.).........216 (L), 217 (L)
Tomonaga–S. (See Koba–Z.).........101, 198, 218, 219 (L)
Utiyama–R. On the Interaction of Meson with the Gravitational Field. 38
Utiyama–R. On the Canonical Transformation in Quantum Theory. 117
Watanabe–S. On Dirac's General Transformation Function.........71
Yamamoto–T. Remarks on the Ferromagnetism of the Semi-Conductors. ....160 (L)
Yamanouchi–T. Elastic Collision Cross-Section of Oxygen Atom for Slow Electrons.........33
Yukawa–H. On the Theory of Elementary Particles. I.........209

Abstracts of the Symposium on the Theory of Elementary Particles. (1946).........92

DATE DUE

GAYLORD

PRINTED IN U.S.A.

3 8198 305 481 283
UNIVERSITY OF ILLINOIS AT CHICAGO